MTH65

Elementary Algebra (Volume 2)

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openstax™

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Table of Contents

PREFACE

Welcome to *Elementary Algebra*, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 25 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax Resources

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Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Elementary Algebra*

Elementary Algebra is designed to meet the scope and sequence requirements of a one-semester elementary algebra course. The book's organization makes it easy to adapt to a variety of course syllabi. The text expands on the fundamental concepts of algebra while addressing the needs of students with diverse backgrounds and learning styles. Each topic builds upon previously developed material to demonstrate the cohesiveness and structure of mathematics.

Coverage and Scope

Elementary Algebra follows a nontraditional approach in its presentation of content. Building on the content in *Prealgebra*, the material is presented as a sequence of small steps so that students gain confidence in their ability to succeed in the course. The order of topics was carefully planned to emphasize the logical progression through the course and to facilitate a thorough understanding of each concept. As new ideas are presented, they are explicitly related to previous topics.

Chapter 1: Foundations
Chapter 1 reviews arithmetic operations with whole numbers, integers, fractions, and decimals, to give the student a solid base that will support their study of algebra.

Chapter 2: Solving Linear Equations and Inequalities
In Chapter 2, students learn to verify a solution of an equation, solve equations using the Subtraction and Addition Properties of Equality, solve equations using the Multiplication and Division Properties of Equality, solve equations with variables and constants on both sides, use a general strategy to solve linear equations, solve equations with fractions or decimals, solve a formula for a specific variable, and solve linear inequalities.

Chapter 3: Math Models
Once students have learned the skills needed to solve equations, they apply these skills in Chapter 3 to solve word and number problems.

Chapter 4: Graphs
Chapter 4 covers the rectangular coordinate system, which is the basis for most consumer graphs. Students learn to plot points on a rectangular coordinate system, graph linear equations in two variables, graph with intercepts,

understand slope of a line, use the slope-intercept form of an equation of a line, find the equation of a line, and create graphs of linear inequalities.

Chapter 5: Systems of Linear Equations
Chapter 5 covers solving systems of equations by graphing, substitution, and elimination; solving applications with systems of equations, solving mixture applications with systems of equations, and graphing systems of linear inequalities.

Chapter 6: Polynomials
In Chapter 6, students learn how to add and subtract polynomials, use multiplication properties of exponents, multiply polynomials, use special products, divide monomials and polynomials, and understand integer exponents and scientific notation.

Chapter 7: Factoring
In Chapter 7, students explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

Chapter 8: Rational Expressions and Equations
In Chapter 8, students work with rational expressions, solve rational equations, and use them to solve problems in a variety of applications.

Chapter 9: Roots and Radical
In Chapter 9, students are introduced to and learn to apply the properties of square roots, and extend these concepts to higher order roots and rational exponents.

Chapter 10: Quadratic Equations
In Chapter 10, students study the properties of quadratic equations, solve and graph them. They also learn how to apply them as models of various situations.

All chapters are broken down into multiple sections, the titles of which can be viewed in the **Table of Contents**.

Key Features and Boxes

Examples Each learning objective is supported by one or more worked examples that demonstrate the problem-solving approaches that students must master. Typically, we include multiple Examples for each learning objective to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity.

All Examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the solution, spelling out the steps along the way. Finally (for select Examples), we show students how to check the solution. Most Examples are written in a two-column format, with explanation on the left and math on the right to mimic the way that instructors "talk through" examples as they write on the board in class.

Be Prepared! Each section, beginning with Section 2.1, starts with a few "Be Prepared!" exercises so that students can determine if they have mastered the prerequisite skills for the section. Reference is made to specific Examples from previous sections so students who need further review can easily find explanations. Answers to these exercises can be found in the supplemental resources that accompany this title.

Try It

The Try It feature includes a pair of exercises that immediately follow an Example, providing the student with an immediate opportunity to solve a similar problem. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try It exercises are located in the Answer Key.

How To

How To feature typically follows the Try It exercises and outlines the series of steps for how to solve the problem in the preceding Example.

Media

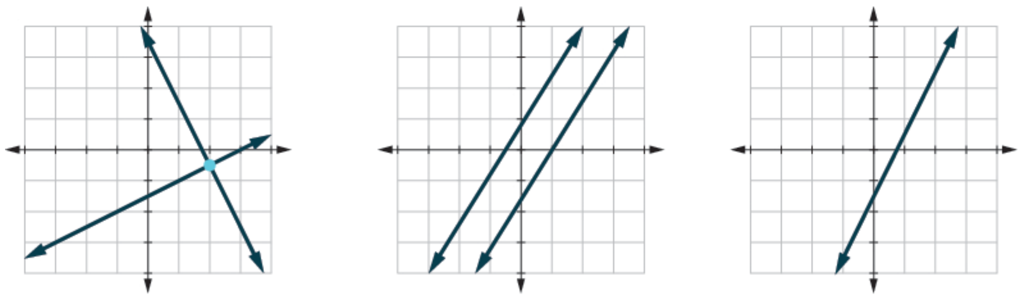
The Media icon appears at the conclusion of each section, just prior to the Self Check. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany *Elementary Algebra*.

Self Check The Self Check includes the learning objectives for the section so that students can self-assess their mastery and make concrete plans to improve.

Art Program

Elementary Algebra contains many figures and illustrations. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions.



Intersecting Parallel Coincident

Section Exercises and Chapter Review

Section Exercises Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. Exercise sets are named *Practice Makes Perfect* to encourage completion of homework assignments.

- Exercises correlate to the learning objectives. This facilitates assignment of personalized study plans based on individual student needs.
- Exercises are carefully sequenced to promote building of skills.
- Values for constants and coefficients were chosen to practice and reinforce arithmetic facts.
- Even and odd-numbered exercises are paired.
- Exercises parallel and extend the text examples and use the same instructions as the examples to help students easily recognize the connection.
- Applications are drawn from many everyday experiences, as well as those traditionally found in college math texts.
- **Everyday Math** highlights practical situations using the concepts from that particular section
- **Writing Exercises** are included in every exercise set to encourage conceptual understanding, critical thinking, and literacy.

Chapter Review Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provide a formal definition for each bold-faced term in the chapter.
- **Key Concepts** summarize the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include practice problems that recall the most important concepts from each section.
- **Practice Test** includes additional problems assessing the most important learning objectives from the chapter.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, manipulative mathematics worksheets, Links to Literacy assignments, and an answer key to Be Prepared Exercises. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

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About the Authors

Senior Contributing Authors

Lynn Marecek and MaryAnne Anthony-Smith have been teaching mathematics at Santa Ana College for many years and have worked together on several projects aimed at improving student learning in developmental math courses. They are the authors of *Strategies for Success: Study Skills for the College Math Student*.

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6 POLYNOMIALS

Figure 6.1 Architects use polynomials to design curved shapes such as this suspension bridge, the Silver Jubilee bridge in Halton, England.

Chapter Outline

Introduction

We have seen that the graphs of linear equations are straight lines. Graphs of other types of equations, called polynomial equations, are curves, like the outline of this suspension bridge. Architects use polynomials to design the shape of a bridge like this and to draw the blueprints for it. Engineers use polynomials to calculate the stress on the bridge's supports to ensure they are strong enough for the intended load. In this chapter, you will explore operations with and properties of polynomials.

6.1 Add and Subtract Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Identify polynomials, monomials, binomials, and trinomials
- Determine the degree of polynomials
- Add and subtract monomials
- Add and subtract polynomials
- Evaluate a polynomial for a given value

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $8x + 3x$.
   If you missed this problem, review **Example 1.24**.
2. Subtract: $(5n + 8) - (2n - 1)$.
   If you missed this problem, review **Example 1.139**.

3. Write in expanded form: a^5.
   If you missed this problem, review **Example 1.14**.

Identify Polynomials, Monomials, Binomials and Trinomials

You have learned that a *term* is a constant or the product of a constant and one or more variables. When it is of the form ax^m, where a is a constant and m is a whole number, it is called a monomial. Some examples of monomial are $8,\ -2x^2,\ 4y^3,\ \text{and}\ 11z^7$.

Monomials

A **monomial** is a term of the form ax^m, where a is a constant and m is a positive whole number.

A monomial, or two or more monomials combined by addition or subtraction, is a polynomial. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a trinomial has exactly three terms. There are no special names for polynomials with more than three terms.

Polynomials

polynomial—A monomial, or two or more monomials combined by addition or subtraction, is a polynomial.

- **monomial**—A polynomial with exactly one term is called a monomial.
- **binomial**—A polynomial with exactly two terms is called a binomial.
- **trinomial**—A polynomial with exactly three terms is called a trinomial.

Here are some examples of polynomials.

Polynomial	$b+1$	$4y^2-7y+2$	$4x^4+x^3+8x^2-9x+1$	
Monomial	14	$8y^2$	$-9x^3y^5$	-13
Binomial	$a+7$	$4b-5$	y^2-16	$3x^3-9x^2$
Trinomial	$x^2-7x+12$	$9y^2+2y-8$	$6m^4-m^3+8m$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

EXAMPLE 6.1

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial.

ⓐ $4y^2-8y-6$ ⓑ $-5a^4b^2$ ⓒ $2x^5-5x^3-9x^2+3x+4$ ⓓ $13-5m^3$ ⓔ q

Solution

	Polynomial	Number of terms	Type
(a)	$4y^2-8y-6$	3	Trinomial
(b)	$-5a^4b^2$	1	Monomial
(c)	$2x^5-5x^3-9x^2+3x+4$	5	Polynomial
(d)	$13-5m^3$	2	Binomial
(e)	q	1	Monomial

TRY IT : : 6.1 Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $5b$ ⓑ $8y^3-7y^2-y-3$ ⓒ $-3x^2-5x+9$ ⓓ $81-4a^2$ ⓔ $-5x^6$

TRY IT : : 6.2 Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial:

ⓐ $27z^3 - 8$ ⓑ $12m^3 - 5m^2 - 2m$ ⓒ $\frac{5}{6}$ ⓓ $8x^4 - 7x^2 - 6x - 5$ ⓔ $-n^4$

Determine the Degree of Polynomials

The degree of a polynomial and the degree of its terms are determined by the exponents of the variable.

A monomial that has no variable, just a constant, is a special case. The degree of a constant is 0—it has no variable.

Degree of a Polynomial

The **degree of a term** is the sum of the exponents of its variables.

The **degree of a constant** is 0.

The **degree of a polynomial** is the highest degree of all its terms.

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms.

Monomial	14	$8y^2$	$-9x^3y^5$	$-13a$
Degree	0	2	8	1

Binomial	$a + 7$	$4b^2 - 5b$	$x^2y^2 - 16$	$3n^3 - 9n^2$
Degree of each term	0 1	2 1	4 0	3 2
Degree of polynomial	1	2	4	3

Trinomial	$x^2 - 7x + 12$	$9a^2 + 6ab + b^2$	$6m^4 - m^3n^2 + 8mn^5$	$z^4 + 3z^2 - 1$
Degree of each term	2 1 0	2 2 2	4 5 6	4 2 0
Degree of polynomial	2	2	6	4

Polynomial	$b + 1$	$4y^2 - 7y + 2$	$4x^4 + x^3 + 8x^2 - 9x + 1$
Degree of each term	1 0	2 1 0	4 3 2 1 0
Degree of polynomial	1	2	4

A polynomial is in **standard form** when the terms of a polynomial are written in descending order of degrees. Get in the habit of writing the term with the highest degree first.

EXAMPLE 6.2

Find the degree of the following polynomials.

ⓐ $10y$ ⓑ $4x^3 - 7x + 5$ ⓒ -15 ⓓ $-8b^2 + 9b - 2$ ⓔ $8xy^2 + 2y$

Solution

ⓐ

	$10y$
The exponent of y is one. $y = y^1$	The degree is 1.

ⓑ

	$4x^3 - 7x + 5$
The highest degree of all the terms is 3.	The degree is 3.

ⓒ

	-15
The degree of a constant is 0.	The degree is 0.

ⓓ

	$-8b^2 + 9b - 2$
The highest degree of all the terms is 2.	The degree is 2.

ⓔ

	$8xy^2 + 2y$
The highest degree of all the terms is 3.	The degree is 3.

TRY IT : : 6.3 Find the degree of the following polynomials:

ⓐ $-15b$ ⓑ $10z^4 + 4z^2 - 5$ ⓒ $12c^5d^4 + 9c^3d^9 - 7$ ⓓ $3x^2y - 4x$ ⓔ -9

TRY IT : : 6.4 Find the degree of the following polynomials:

ⓐ 52 ⓑ $a^4b - 17a^4$ ⓒ $5x + 6y + 2z$ ⓓ $3x^2 - 5x + 7$ ⓔ $-a^3$

Add and Subtract Monomials

You have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponent. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficient.

EXAMPLE 6.3

Add: $25y^2 + 15y^2$.

Solution

	$25y^2 + 15y^2$
Combine like terms.	$40y^2$

TRY IT : : 6.5 Add: $12q^2 + 9q^2$.

TRY IT : : 6.6 Add: $-15c^2 + 8c^2$.

EXAMPLE 6.4

Subtract: $16p - (-7p)$.

Solution

	$16p - (-7p)$
Combine like terms.	$23p$

TRY IT : : 6.7 Subtract: $8m - (-5m)$.

TRY IT : : 6.8 Subtract: $-15z^3 - \left(-5z^3\right)$.

Remember that like terms must have the same variables with the same exponents.

EXAMPLE 6.5

Simplify: $c^2 + 7d^2 - 6c^2$.

Solution

	$c^2 + 7d^2 - 6c^2$
Combine like terms.	$-5c^2 + 7d^2$

TRY IT : : 6.9 Add: $8y^2 + 3z^2 - 3y^2$.

TRY IT : : 6.10 Add: $3m^2 + n^2 - 7m^2$.

EXAMPLE 6.6

Simplify: $u^2 v + 5u^2 - 3v^2$.

Solution

	$u^2 v + 5u^2 - 3v^2$
There are no like terms to combine.	$u^2 v + 5u^2 - 3v^2$

TRY IT : : 6.11 Simplify: $m^2 n^2 - 8m^2 + 4n^2$.

TRY IT : : 6.12 Simplify: $pq^2 - 6p - 5q^2$.

Add and Subtract Polynomials

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

EXAMPLE 6.7

Find the sum: $(5y^2 - 3y + 15) + (3y^2 - 4y - 11)$.

Solution

Identify like terms.	$(5y^2 - 3y + 15) + (3y^2 - 4y - 11)$
Rearrange to get the like terms together.	$5y^2 + 3y^2 - 3y - 4y + 15 - 11$
Combine like terms.	$8y^2 - 7y + 4$

TRY IT : : 6.13 Find the sum: $(7x^2 - 4x + 5) + (x^2 - 7x + 3)$.

TRY IT : : 6.14 Find the sum: $(14y^2 + 6y - 4) + (3y^2 + 8y + 5)$.

EXAMPLE 6.8

Find the difference: $\left(9w^2 - 7w + 5\right) - \left(2w^2 - 4\right)$.

Solution

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Distribute and identify like terms.	$9w^2 - 7w + 5 - 2w^2 + 4$
Rearrange the terms.	$9w^2 - 2w^2 - 7w + 5 + 4$
Combine like terms.	$7w^2 - 7w + 9$

TRY IT : : 6.15 Find the difference: $\left(8x^2 + 3x - 19\right) - \left(7x^2 - 14\right)$.

TRY IT : : 6.16 Find the difference: $\left(9b^2 - 5b - 4\right) - \left(3b^2 - 5b - 7\right)$.

EXAMPLE 6.9

Subtract: $\left(c^2 - 4c + 7\right)$ from $\left(7c^2 - 5c + 3\right)$.

Solution

	Subtract $(c^2 - 4c + 7)$ from $(7c^2 - 5c + 3)$.
	$(7c^2 - 5c + 3) - (c^2 - 4c + 7)$
Distribute and identify like terms.	$7c^2 - 5c + 3 - c^2 + 4c - 7$
Rearrange the terms.	$7c^2 - c^2 - 5c + 4c + 3 - 7$
Combine like terms.	$6c^2 - c - 4$

TRY IT : : 6.17 Subtract: $\left(5z^2 - 6z - 2\right)$ from $\left(7z^2 + 6z - 4\right)$.

TRY IT : : 6.18 Subtract: $\left(x^2 - 5x - 8\right)$ from $\left(6x^2 + 9x - 1\right)$.

EXAMPLE 6.10

Find the sum: $\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$.

Solution

	$\left(u^2 - 6uv + 5v^2\right) + \left(3u^2 + 2uv\right)$
Distribute.	$u^2 - 6uv + 5v^2 + 3u^2 + 2uv$
Rearrange the terms, to put like terms together.	$u^2 + 3u^2 - 6uv + 2uv + 5v^2$
Combine like terms.	$4u^2 - 4uv + 5v^2$

TRY IT : : 6.19 Find the sum: $(3x^2 - 4xy + 5y^2) + (2x^2 - xy)$.

TRY IT : : 6.20 Find the sum: $(2x^2 - 3xy - 2y^2) + (5x^2 - 3xy)$.

EXAMPLE 6.11

Find the difference: $(p^2 + q^2) - (p^2 + 10pq - 2q^2)$.

Solution

	$(p^2 + q^2) - (p^2 + 10pq - 2q^2)$
Distribute.	$p^2 + q^2 - p^2 - 10pq + 2q^2$
Rearrange the terms, to put like terms together.	$p^2 - p^2 - 10pq + q^2 + 2q^2$
Combine like terms.	$-10pq^2 + 3q^2$

TRY IT : : 6.21 Find the difference: $(a^2 + b^2) - (a^2 + 5ab - 6b^2)$.

TRY IT : : 6.22 Find the difference: $(m^2 + n^2) - (m^2 - 7mn - 3n^2)$.

EXAMPLE 6.12

Simplify: $(a^3 - a^2 b) - (ab^2 + b^3) + (a^2 b + ab^2)$.

Solution

	$(a^3 - a^2 b) - (ab^2 + b^3) + (a^2 b + ab^2)$
Distribute.	$a^3 - a^2 b - ab^2 - b^3 + a^2 b + ab^2$
Rearrange the terms, to put like terms together.	$a^3 - a^2 b + a^2 b - ab^2 + ab^2 - b^3$
Combine like terms.	$a^3 - b^3$

TRY IT : : 6.23 Simplify: $(x^3 - x^2 y) - (xy^2 + y^3) + (x^2 y + xy^2)$.

TRY IT : : 6.24 Simplify: $(p^3 - p^2 q) + (pq^2 + q^3) - (p^2 q + pq^2)$.

Evaluate a Polynomial for a Given Value

We have already learned how to evaluate expressions. Since polynomials are expressions, we'll follow the same procedures to evaluate a polynomial. We will substitute the given value for the variable and then simplify using the order of operations.

EXAMPLE 6.13

Evaluate $5x^2 - 8x + 4$ when

ⓐ $x = 4$ ⓑ $x = -2$ ⓒ $x = 0$

Solution

ⓐ $x = 4$

	$5x^2 - 8x + 4$
Substitute 4 for x.	$5(4)^2 - 8(4) + 4$
Simplify the exponents.	$5 \cdot 16 - 8(4) + 4$
Multiply.	$80 - 32 + 4$
Simplify.	52

ⓑ $x = -2$

	$5x^2 - 8x + 4$
Substitute −2 for x.	$5(-2)^2 - 8(-2) + 4$
Simplify the exponents.	$5 \cdot 4 - 8(-2) + 4$
Multiply.	$20 + 16 + 4$
Simplify.	40

ⓒ $x = 0$

	$5x^2 - 8x + 4$
Substitute 0 for x.	$5(0)^2 - 8(0) + 4$
Simplify the exponents.	$5 \cdot 0 - 8(0) + 4$
Multiply.	$0 + 0 + 4$
Simplify.	4

TRY IT : : 6.25 Evaluate: $3x^2 + 2x - 15$ when

ⓐ $x = 3$ ⓑ $x = -5$ ⓒ $x = 0$

TRY IT : : 6.26 Evaluate: $5z^2 - z - 4$ when

ⓐ $z = -2$ ⓑ $z = 0$ ⓒ $z = 2$

EXAMPLE 6.14

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250 foot tall building. Find the height after $t = 2$ seconds.

Solution

	$-16t^2 + 250$
Substitute $t = 2$.	$-16(2)^2 + 250$
Simplify.	$-16 \cdot 4 + 250$
Simplify.	$-64 + 250$
Simplify.	186
	After 2 seconds the height of the ball is 186 feet.

TRY IT : : 6.27

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 0$ seconds.

TRY IT : : 6.28

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 3$ seconds.

EXAMPLE 6.15

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 4$ feet and $y = 6$ feet.

Solution

	$6x^2 + 15xy$
Substitute $x = 4$, $y = 6$.	$6(4)^2 + 15(4)(6)$
Simplify.	$6 \cdot 16 + 15(4)(6)$
Simplify.	$96 + 360$
Simplify.	456
	The cost of producing the box is $456.

TRY IT : : 6.29

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 6$ feet and $y = 4$ feet.

TRY IT : : 6.30

The polynomial $6x^2 + 15xy$ gives the cost, in dollars, of producing a rectangular container whose top and bottom are squares with side *x* feet and sides of height *y* feet. Find the cost of producing a box with $x = 5$ feet and $y = 8$ feet.

MEDIA : :

Access these online resources for additional instruction and practice with adding and subtracting polynomials.

- **Add and Subtract Polynomials 1 (https://openstax.org/l/25Addsubtrpoly1)**
- **Add and Subtract Polynomials 2 (https://openstax.org/l/25Addsubtrpoly2)**
- **Add and Subtract Polynomial 3 (https://openstax.org/l/25Addsubtrpoly3)**
- **Add and Subtract Polynomial 4 (https://openstax.org/l/25Addsubtrpoly4)**

6.1 EXERCISES

Practice Makes Perfect

Identify Polynomials, Monomials, Binomials, and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

1.
ⓐ $81b^5 - 24b^3 + 1$
ⓑ $5c^3 + 11c^2 - c - 8$
ⓒ $\frac{14}{15}y + \frac{1}{7}$
ⓓ 5
ⓔ $4y + 17$

2.
ⓐ $x^2 - y^2$
ⓑ $-13c^4$
ⓒ $x^2 + 5x - 7$
ⓓ $x^2y^2 - 2xy + 8$
ⓔ 19

3.
ⓐ $8 - 3x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $81b^5 - 24b^3 + 1$
ⓔ -18

4.
ⓐ $11y^2$
ⓑ -73
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $4y + 17$
ⓔ $5c^3 + 11c^2 - c - 8$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

5.
ⓐ $6a^2 + 12a + 14$
ⓑ $18xy^2z$
ⓒ $5x + 2$
ⓓ $y^3 - 8y^2 + 2y - 16$
ⓔ -24

6.
ⓐ $9y^3 - 10y^2 + 2y - 6$
ⓑ $-12p^4$
ⓒ $a^2 + 9a + 18$
ⓓ $20x^2y^2 - 10a^2b^2 + 30$
ⓔ 17

7.
ⓐ $14 - 29x$
ⓑ $z^2 - 5z - 6$
ⓒ $y^3 - 8y^2 + 2y - 16$
ⓓ $23ab^2 - 14$
ⓔ -3

8.
ⓐ $62y^2$
ⓑ 15
ⓒ $6x^2 - 3xy + 4x - 2y + y^2$
ⓓ $10 - 9x$
ⓔ $m^4 + 4m^3 + 6m^2 + 4m + 1$

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

9. $7x^2 + 5x^2$

10. $4y^3 + 6y^3$

11. $-12w + 18w$

12. $-3m + 9m$

13. $4a - 9a$

14. $-y - 5y$

15. $28x - (-12x)$

16. $13z - (-4z)$

17. $-5b - 17b$

18. $-10x - 35x$

19. $12a + 5b - 22a$

20. $14x - 3y - 13x$

21. $2a^2 + b^2 - 6a^2$

22. $5u^2 + 4v^2 - 6u^2$

23. $xy^2 - 5x - 5y^2$

24. $pq^2 - 4p - 3q^2$

25. $a^2 b - 4a - 5ab^2$

26. $x^2 y - 3x + 7xy^2$

27. $12a + 8b$

28. $19y + 5z$

29. Add: $4a,\ -3b,\ -8a$

30. Add: $4x,\ 3y,\ -3x$

31. Subtract $5x^6$ from $-12x^6$.

32. Subtract $2p^4$ from $-7p^4$.

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

33. $(5y^2 + 12y + 4) + (6y^2 - 8y + 7)$

34. $(4y^2 + 10y + 3) + (8y^2 - 6y + 5)$

35. $(x^2 + 6x + 8) + (-4x^2 + 11x - 9)$

36. $(y^2 + 9y + 4) + (-2y^2 - 5y - 1)$

37. $(8x^2 - 5x + 2) + (3x^2 + 3)$

38. $(7x^2 - 9x + 2) + (6x^2 - 4)$

39. $(5a^2 + 8) + (a^2 - 4a - 9)$

40. $(p^2 - 6p - 18) + (2p^2 + 11)$

41. $(4m^2 - 6m - 3) - (2m^2 + m - 7)$

42. $(3b^2 - 4b + 1) - (5b^2 - b - 2)$

43. $(a^2 + 8a + 5) - (a^2 - 3a + 2)$

44. $(b^2 - 7b + 5) - (b^2 - 2b + 9)$

45. $(12s^2 - 15s) - (s - 9)$

46. $(10r^2 - 20r) - (r - 8)$

47. Subtract $\left(9x^2 + 2\right)$ from $\left(12x^2 - x + 6\right)$.

48. Subtract $\left(5y^2 - y + 12\right)$ from $\left(10y^2 - 8y - 20\right)$.

49. Subtract $\left(7w^2 - 4w + 2\right)$ from $\left(8w^2 - w + 6\right)$.

50. Subtract $\left(5x^2 - x + 12\right)$ from $\left(9x^2 - 6x - 20\right)$.

51. Find the sum of $\left(2p^3 - 8\right)$ and $\left(p^2 + 9p + 18\right)$.

52. Find the sum of $\left(q^2 + 4q + 13\right)$ and $\left(7q^3 - 3\right)$.

53. Find the sum of $\left(8a^3 - 8a\right)$ and $\left(a^2 + 6a + 12\right)$.

54. Find the sum of $\left(b^2 + 5b + 13\right)$ and $\left(4b^3 - 6\right)$.

55. Find the difference of $\left(w^2 + w - 42\right)$ and $\left(w^2 - 10w + 24\right)$.

56. Find the difference of $\left(z^2 - 3z - 18\right)$ and $\left(z^2 + 5z - 20\right)$.

57. Find the difference of $\left(c^2 + 4c - 33\right)$ and $\left(c^2 - 8c + 12\right)$.

58. Find the difference of $\left(t^2 - 5t - 15\right)$ and $\left(t^2 + 4t - 17\right)$.

59. $\left(7x^2 - 2xy + 6y^2\right) + \left(3x^2 - 5xy\right)$

60. $\left(-5x^2 - 4xy - 3y^2\right) + \left(2x^2 - 7xy\right)$

61. $\left(7m^2 + mn - 8n^2\right) + \left(3m^2 + 2mn\right)$

62. $\left(2r^2 - 3rs - 2s^2\right) + \left(5r^2 - 3rs\right)$

63. $\left(a^2 - b^2\right) - \left(a^2 + 3ab - 4b^2\right)$

64. $\left(m^2 + 2n^2\right) - \left(m^2 - 8mn - n^2\right)$

65. $\left(u^2 - v^2\right) - \left(u^2 - 4uv - 3v^2\right)$

66. $\left(j^2 - k^2\right) - \left(j^2 - 8jk - 5k^2\right)$

67. $\left(p^3 - 3p^2q\right) + \left(2pq^2 + 4q^3\right) - \left(3p^2q + pq^2\right)$

68. $\left(a^3 - 2a^2b\right) + \left(ab^2 + b^3\right) - \left(3a^2b + 4ab^2\right)$

69. $\left(x^3 - x^2y\right) - \left(4xy^2 - y^3\right) + \left(3x^2y - xy^2\right)$

70. $\left(x^3 - 2x^2y\right) - \left(xy^2 - 3y^3\right) - \left(x^2y - 4xy^2\right)$

Evaluate a Polynomial for a Given Value

In the following exercises, evaluate each polynomial for the given value.

71. Evaluate $8y^2 - 3y + 2$ when:

ⓐ $y = 5$

ⓑ $y = -2$

ⓒ $y = 0$

72. Evaluate $5y^2 - y - 7$ when:

ⓐ $y = -4$

ⓑ $y = 1$

ⓒ $y = 0$

73. Evaluate $4 - 36x$ when:

ⓐ $x = 3$

ⓑ $x = 0$

ⓒ $x = -1$

74. Evaluate $16 - 36x^2$ when:

ⓐ $x = -1$

ⓑ $x = 0$

ⓒ $x = 2$

75. A painter drops a brush from a platform 75 feet high. The polynomial $-16t^2 + 75$ gives the height of the brush t seconds after it was dropped. Find the height after $t = 2$ seconds.

76. A girl drops a ball off a cliff into the ocean. The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall cliff. Find the height after $t = 2$ seconds.

77. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 60$ dollars.

78. A manufacturer of the latest basketball shoes has found that the revenue received from selling the shoes at a cost of *p* dollars each is given by the polynomial $-4p^2 + 420p$. Find the revenue received when $p = 90$ dollars.

Everyday Math

79. Fuel Efficiency The fuel efficiency (in miles per gallon) of a car going at a speed of x miles per hour is given by the polynomial $-\frac{1}{150}x^2 + \frac{1}{3}x$. Find the fuel efficiency when $x = 30\text{ mph}$.

80. Stopping Distance The number of feet it takes for a car traveling at x miles per hour to stop on dry, level concrete is given by the polynomial $0.06x^2 + 1.1x$. Find the stopping distance when $x = 40\text{ mph}$.

81. Rental Cost The cost to rent a rug cleaner for d days is given by the polynomial $5.50d + 25$. Find the cost to rent the cleaner for 6 days.

82. Height of Projectile The height (in feet) of an object projected upward is given by the polynomial $-16t^2 + 60t + 90$ where t represents time in seconds. Find the height after $t = 2.5$ seconds.

83. Temperature Conversion The temperature in degrees Fahrenheit is given by the polynomial $\frac{9}{5}c + 32$ where c represents the temperature in degrees Celsius. Find the temperature in degrees Fahrenheit when $c = 65°$.

Writing Exercises

84. Using your own words, explain the difference between a monomial, a binomial, and a trinomial.

85. Using your own words, explain the difference between a polynomial with five terms and a polynomial with a degree of 5.

86. Ariana thinks the sum $6y^2 + 5y^4$ is $11y^6$. What is wrong with her reasoning?

87. Jonathan thinks that $\frac{1}{3}$ and $\frac{1}{x}$ are both monomials. What is wrong with his reasoning?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
identify polynomials, monomials, binomials, and trinomials.			
determine the degree of polynomials.			
add and subtract monomials.			
add and subtract polynomials.			
evaluate a polynomial for a given value.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

6.2 Use Multiplication Properties of Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with exponents
- Simplify expressions using the Product Property for Exponents
- Simplify expressions using the Power Property for Exponents
- Simplify expressions using the Product to a Power Property
- Simplify expressions by applying several properties
- Multiply monomials

Be Prepared!

Before you get started, take this readiness quiz.

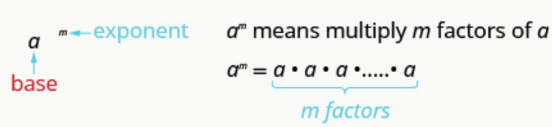
1. Simplify: $\frac{3}{4} \cdot \frac{3}{4}$.
   If you missed this problem, review **Example 1.68**.
2. Simplify: $(-2)(-2)(-2)$.
   If you missed this problem, review **Example 1.50**.

Simplify Expressions with Exponents

Remember that an exponent indicates repeated multiplication of the same quantity. For example, 2^4 means to multiply 2 by itself 4 times, so 2^4 means $2 \cdot 2 \cdot 2 \cdot 2$.

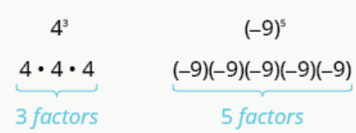
Let's review the vocabulary for expressions with exponents.

Exponential Notation



This is read a to the m^{th} power.

In the expression a^m, the *exponent* m tells us how many times we use the *base* a as a factor.



Before we begin working with variable expressions containing exponents, let's simplify a few expressions involving only numbers.

EXAMPLE 6.16

Simplify: ⓐ 4^3 ⓑ 7^1 ⓒ $\left(\frac{5}{6}\right)^2$ ⓓ $(0.63)^2$.

Solution

ⓐ

	4^3
Multiply three factors of 4.	$4 \cdot 4 \cdot 4$
Simplify.	64

ⓑ

	7^1
Multiply one factor of 7.	7

ⓒ

	$\left(\frac{5}{6}\right)^2$
Multiply two factors.	$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)$
Simplify.	$\frac{25}{36}$

ⓓ

	$(0.63)^2$
Multiply two factors.	$(0.63)(0.63)$
Simplify.	0.3969

TRY IT : : 6.31 Simplify: ⓐ 6^3 ⓑ 15^1 ⓒ $\left(\frac{3}{7}\right)^2$ ⓓ $(0.43)^2$.

TRY IT : : 6.32 Simplify: ⓐ 2^5 ⓑ 21^1 ⓒ $\left(\frac{2}{5}\right)^3$ ⓓ $(0.218)^2$.

EXAMPLE 6.17

Simplify: ⓐ $(-5)^4$ ⓑ -5^4.

Solution

ⓐ

	$(-5)^4$
Multiply four factors of -5.	$(-5)(-5)(-5)(-5)$
Simplify.	625

ⓑ

	-5^4
Multiply four factors of 5.	$-(5 \cdot 5 \cdot 5 \cdot 5)$
Simplify.	-625

TRY IT : : 6.33 Simplify: ⓐ $(-3)^4$ ⓑ -3^4.

TRY IT : : 6.34 Simplify: ⓐ $(-13)^2$ ⓑ -13^2.

Notice the similarities and differences in **Example 6.17**ⓐ and **Example 6.17**ⓑ! Why are the answers different? As we follow the order of operations in part ⓐ the parentheses tell us to raise the (-5) to the 4th power. In part ⓑ we raise just the 5 to the 4th power and then take the opposite.

Simplify Expressions Using the Product Property for Exponents

You have seen that when you combine like terms by adding and subtracting, you need to have the same base with the same exponent. But when you multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

We'll derive the properties of exponents by looking for patterns in several examples.

First, we will look at an example that leads to the Product Property.

	$x^2 \cdot x^3$
What does this mean? How many factors altogether?	$\underbrace{\underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x \cdot x}_{3\ factors}}_{5\ factors}$
So, we have	x^5
Notice that 5 is the sum of the exponents, 2 and 3.	$x^2 \cdot x^3$ is x^{2+3}, or x^5

We write:

$$x^2 \cdot x^3$$
$$x^{2+3}$$
$$x^5$$

The base stayed the same and we added the exponents. This leads to the **Product Property for Exponents**.

Product Property for Exponents

If a is a real number, and m and n are counting numbers, then

$$a^m \cdot a^n = a^{m+n}$$

To multiply with like bases, add the exponents.

An example with numbers helps to verify this property.

$$2^2 \cdot 2^3 \overset{?}{=} 2^{2+3}$$
$$4 \cdot 8 \overset{?}{=} 2^5$$
$$32 = 32 \checkmark$$

EXAMPLE 6.18

Simplify: $y^5 \cdot y^6$.

Solution

	$y^5 \cdot y^6$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	y^{5+6}
Simplify.	y^{11}

TRY IT : : 6.35 Simplify: $b^9 \cdot b^8$.

TRY IT : : 6.36 Simplify: $x^{12} \cdot x^4$.

EXAMPLE 6.19

Simplify: ⓐ $2^5 \cdot 2^9$ ⓑ $3 \cdot 3^4$.

Solution

ⓐ

	$2^5 \cdot 2^9$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	2^{5+9}
Simplify.	2^{14}

ⓑ

	$3 \cdot 3^4$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	3^{1+4}
Simplify.	3^5

TRY IT : : 6.37 Simplify: ⓐ $5 \cdot 5^5$ ⓑ $4^9 \cdot 4^9$.

TRY IT : : 6.38 Simplify: ⓐ $7^6 \cdot 7^8$ ⓑ $10 \cdot 10^{10}$.

EXAMPLE 6.20

Simplify: ⓐ $a^7 \cdot a$ ⓑ $x^{27} \cdot x^{13}$.

Solution

ⓐ

	$a^7 \cdot a$
Rewrite, $a = a^1$.	$a^7 \cdot a^1$
Use the product property, $a^m \cdot a^n = a^{m+n}$.	a^{7+1}
Simplify.	a^8

ⓑ

	$x^{27} \cdot x^{13}$
Notice, the bases are the same, so add the exponents.	x^{27+13}
Simplify.	x^{40}

TRY IT : : 6.39 Simplify: ⓐ $p^5 \cdot p$ ⓑ $y^{14} \cdot y^{29}$.

TRY IT : : 6.40 Simplify: ⓐ $z \cdot z^7$ ⓑ $b^{15} \cdot b^{34}$.

We can extend the Product Property for Exponents to more than two factors.

EXAMPLE 6.21

Simplify: $d^4 \cdot d^5 \cdot d^2$.

Solution

	$d^4 \cdot d^5 \cdot d^2$
Add the exponents, since bases are the same.	d^{4+5+2}
Simplify.	d^{11}

TRY IT : : 6.41 Simplify: $x^6 \cdot x^4 \cdot x^8$.

TRY IT : : 6.42 Simplify: $b^5 \cdot b^9 \cdot b^5$.

Simplify Expressions Using the Power Property for Exponents

Now let’s look at an exponential expression that contains a power raised to a power. See if you can discover a general property.

	$(x^2)^3$
What does this mean? How many factors altogether?	$x^2 \cdot x^2 \cdot x^2$ $\underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\ factors}$ 6 factors
So we have	x^6
Notice that 6 is the product of the exponents, 2 and 3.	$(x^2)^3$ is $x^{2 \cdot 3}$ or x^6

We write:

$$\left(x^2\right)^3$$
$$x^{2 \cdot 3}$$
$$x^6$$

We multiplied the exponents. This leads to the **Power Property for Exponents.**

Power Property for Exponents

If a is a real number, and m and n are whole numbers, then

$$(a^m)^n = a^{m \cdot n}$$

To raise a power to a power, multiply the exponents.

An example with numbers helps to verify this property.

$$\begin{aligned} \left(3^2\right)^3 &\stackrel{?}{=} 3^{2 \cdot 3} \\ (9)^3 &\stackrel{?}{=} 3^6 \\ 729 &= 729 \checkmark \end{aligned}$$

EXAMPLE 6.22

Simplify: ⓐ $(y^5)^9$ ⓑ $(4^4)^7$.

Solution

ⓐ

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{m \cdot n}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

ⓑ

	$(4^4)^7$
Use the power property.	$4^{4\cdot7}$
Simplify.	4^{28}

> **TRY IT : :** 6.43 Simplify: ⓐ $(b^7)^5$ ⓑ $(5^4)^3$.

> **TRY IT : :** 6.44 Simplify: ⓐ $(z^6)^9$ ⓑ $(3^7)^7$.

Simplify Expressions Using the Product to a Power Property

We will now look at an expression containing a product that is raised to a power. Can you find this pattern?

	$(2x)^3$
What does this mean?	$2x\cdot 2x\cdot 2x$
We group the like factors together.	$2\cdot2\cdot2\cdot x\cdot x\cdot x$
How many factors of 2 and of x?	$2^3\cdot x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3\cdot x^3$.

We write:	$(2x)^3$
	$2^3\cdot x^3$

The exponent applies to each of the factors! This leads to the **Product to a Power Property for Exponents.**

Product to a Power Property for Exponents

If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.

An example with numbers helps to verify this property:

$$\begin{aligned}(2\cdot3)^2 &\stackrel{?}{=} 2^2\cdot3^2\\ 6^2 &\stackrel{?}{=} 4\cdot9\\ 36 &= 36\checkmark\end{aligned}$$

EXAMPLE 6.23

Simplify: ⓐ $(-9d)^2$ ⓑ $(3mn)^3$.

Solution

ⓐ

	$(-9d)^2$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(-9)^2 d^2$
Simplify.	$81d^2$

ⓑ

	$(3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$(3)^3 m^3 n^3$
Simplify.	$27m^3n^3$

TRY IT : : 6.45 Simplify: ⓐ $(-12y)^2$ ⓑ $(2wx)^5$.

TRY IT : : 6.46 Simplify: ⓐ $(5wx)^3$ ⓑ $(-3y)^3$.

Simplify Expressions by Applying Several Properties

We now have three properties for multiplying expressions with exponents. Let's summarize them and then we'll do some examples that use more than one of the properties.

Properties of Exponents

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$

All exponent properties hold true for any real numbers m and n. Right now, we only use whole number exponents.

EXAMPLE 6.24

Simplify: ⓐ $(y^3)^6 (y^5)^4$ ⓑ $\left(-6x^4 y^5\right)^2$.

Solution

ⓐ

	$(y^3)^6 (y^5)^4$
Use the Power Property.	$y^{15} \cdot y^{20}$
Add the exponents.	y^{35}

ⓑ

	$\left(-6x^4 y^5\right)^2$
Use the Product to a Power Property.	$(-6)^2 (x^4)^2 (y^5)^2$
Use the Power Property.	$(-6)^2 (x^8)(y^{10})$
Simplify.	$36x^8 y^{10}$

TRY IT : : 6.47 Simplify: ⓐ $(a^4)^5 (a^7)^4$ ⓑ $\left(-2c^4 d^2\right)^3$.

TRY IT : : 6.48 Simplify: ⓐ $\left(-3x^6 y^7\right)^4$ ⓑ $(q^4)^5 (q^3)^3$.

EXAMPLE 6.25

Simplify: ⓐ $(5m)^2(3m^3)$ ⓑ $(3x^2y)^4(2xy^2)^3$.

Solution

ⓐ

	$(5m)^2(3m^3)$
Raise $5m$ to the second power.	$5^2m^2 \cdot 3m^3$
Simplify.	$25m^2 \cdot 3m^3$
Use the Commutative Property.	$25 \cdot 3 \cdot m^2 \cdot m^3$
Multiply the constants and add the exponents.	$75m^5$

ⓑ

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property.	$(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$81 \cdot 8 \cdot x^8 \cdot x^3 \cdot y^4 \cdot y^6$
Multiply the constants and add the exponents.	$648x^{11}y^{10}$

TRY IT : : 6.49 Simplify: ⓐ $(5n)^2(3n^{10})$ ⓑ $(c^4d^2)^5(3cd^5)^4$.

TRY IT : : 6.50 Simplify: ⓐ $(a^3b^2)^6(4ab^3)^4$ ⓑ $(2x)^3(5x^7)$.

Multiply Monomials

Since a monomial is an algebraic expression, we can use the properties of exponents to multiply monomials.

EXAMPLE 6.26

Multiply: $\left(3x^2\right)\left(-4x^3\right)$.

Solution

	$\left(3x^2\right)\left(-4x^3\right)$
Use the Commutative Property to rearrange the terms.	$3 \cdot (-4) \cdot x^2 \cdot x^3$
Multiply.	$-12x^5$

TRY IT : : 6.51 Multiply: $\left(5y^7\right)\left(-7y^4\right)$.

TRY IT : : 6.52 Multiply: $\left(-6b^4\right)\left(-9b^5\right)$.

EXAMPLE 6.27

Multiply: $\left(\frac{5}{6}x^3y\right)\left(12xy^2\right)$.

Solution

	$\left(\frac{5}{6}x^3 y\right)\left(12xy^2\right)$
Use the Commutative Property to rearrange the terms.	$\frac{5}{6} \cdot 12 \cdot x^3 \cdot x \cdot y \cdot y^2$
Multiply.	$10x^4 y^3$

TRY IT : : 6.53 Multiply: $\left(\frac{2}{5}a^4 b^3\right)\left(15ab^3\right)$.

TRY IT : : 6.54 Multiply: $\left(\frac{2}{3}r^5 s\right)\left(12r^6 s^7\right)$.

MEDIA : :
Access these online resources for additional instruction and practice with using multiplication properties of exponents:

- **Multiplication Properties of Exponents (https://openstax.org/l/25MultiPropExp)**

6.2 EXERCISES

Practice Makes Perfect

Simplify Expressions with Exponents

In the following exercises, simplify each expression with exponents.

88.
ⓐ 3^5
ⓑ 9^1
ⓒ $\left(\frac{1}{3}\right)^2$
ⓓ $(0.2)^4$

89.
ⓐ 10^4
ⓑ 17^1
ⓒ $\left(\frac{2}{9}\right)^2$
ⓓ $(0.5)^3$

90.
ⓐ 2^6
ⓑ 14^1
ⓒ $\left(\frac{2}{5}\right)^3$
ⓓ $(0.7)^2$

91.
ⓐ 8^3
ⓑ 8^1
ⓒ $\left(\frac{3}{4}\right)^3$
ⓓ $(0.4)^3$

92.
ⓐ $(-6)^4$
ⓑ -6^4

93.
ⓐ $(-2)^6$
ⓑ -2^6

94.
ⓐ $-\left(\frac{1}{4}\right)^4$
ⓑ $\left(-\frac{1}{4}\right)^4$

95.
ⓐ $-\left(\frac{2}{3}\right)^2$
ⓑ $\left(-\frac{2}{3}\right)^2$

96.
ⓐ -0.5^2
ⓑ $(-0.5)^2$

97.
ⓐ -0.1^4
ⓑ $(-0.1)^4$

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression using the Product Property for Exponents.

98. $d^3 \cdot d^6$

99. $x^4 \cdot x^2$

100. $n^{19} \cdot n^{12}$

101. $q^{27} \cdot q^{15}$

102. ⓐ $4^5 \cdot 4^9$ ⓑ $8^9 \cdot 8$

103. ⓐ $3^{10} \cdot 3^6$ ⓑ $5 \cdot 5^4$

104. ⓐ $y \cdot y^3$ ⓑ $z^{25} \cdot z^8$

105. ⓐ $w^5 \cdot w$ ⓑ $u^{41} \cdot u^{53}$

106. $w \cdot w^2 \cdot w^3$

107. $y \cdot y^3 \cdot y^5$

108. $a^4 \cdot a^3 \cdot a^9$

109. $c^5 \cdot c^{11} \cdot c^2$

110. $m^x \cdot m^3$

111. $n^y \cdot n^2$

112. $y^a \cdot y^b$

113. $x^p \cdot x^q$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression using the Power Property for Exponents.

114. ⓐ $(m^4)^2$ ⓑ $(10^3)^6$

115. ⓐ $(b^2)^7$ ⓑ $(3^8)^2$

116. ⓐ $\left(y^3\right)^x$ ⓑ $(5^x)^y$

117. ⓐ $\left(x^2\right)^y$ ⓑ $(7^a)^b$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression using the Product to a Power Property.

118. ⓐ $(6a)^2$ ⓑ $(3xy)^2$

119. ⓐ $(5x)^2$ ⓑ $(4ab)^2$

120. ⓐ $(-4m)^3$ ⓑ $(5ab)^3$

121. ⓐ $(-7n)^3$ ⓑ $(3xyz)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

122.
ⓐ $\left(y^2\right)^4 \cdot \left(y^3\right)^2$
ⓑ $(10a^2b)^3$

123.
ⓐ $\left(w^4\right)^3 \cdot \left(w^5\right)^2$
ⓑ $(2xy^4)^5$

124.
ⓐ $(-2r^3s^2)^4$
ⓑ $\left(m^5\right)^3 \cdot \left(m^9\right)^4$

125.
ⓐ $(-10q^2p^4)^3$
ⓑ $\left(n^3\right)^{10} \cdot \left(n^5\right)^2$

126.
ⓐ $(3x)^2(5x)$
ⓑ $(5t^2)^3(3t)^2$

127.
ⓐ $(2y)^3(6y)$
ⓑ $(10k^4)^3(5k^6)^2$

128.
ⓐ $(5a)^2(2a)^3$
ⓑ $\left(\frac{1}{2}y^2\right)^3\left(\frac{2}{3}y\right)^2$

129.
ⓐ $(4b)^2(3b)^3$
ⓑ $\left(\frac{1}{2}j^2\right)^5\left(\frac{2}{5}j^3\right)^2$

130.
ⓐ $\left(\frac{2}{5}x^2y\right)^3$
ⓑ $\left(\frac{8}{9}xy^4\right)^2$

131.
ⓐ $(2r^2)^3(4r)^2$
ⓑ $(3x^3)^3(x^5)^4$

132.
ⓐ $(m^2n)^2(2mn^5)^4$
ⓑ $(3pq^4)^2(6p^6q)^2$

Multiply Monomials

In the following exercises, multiply the monomials.

133. $\left(6y^7\right)\left(-3y^4\right)$

134. $\left(-10x^5\right)\left(-3x^3\right)$

135. $(-8u^6)(-9u)$

136. $(-6c^4)(-12c)$

137. $\left(\frac{1}{5}f^8\right)(20f^3)$

138. $\left(\frac{1}{4}d^5\right)(36d^2)$

139. $(4a^3b)(9a^2b^6)$

140. $(6m^4n^3)(7mn^5)$

141. $\left(\frac{4}{7}rs^2\right)\left(14rs^3\right)$

142. $\left(\frac{5}{8}x^3y\right)\left(24x^5y\right)$

143. $\left(\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$

144. $\left(\frac{3}{5}m^3n^2\right)\left(\frac{5}{9}m^2n^3\right)$

Mixed Practice

In the following exercises, simplify each expression.

145. $\left(x^2\right)^4 \cdot \left(x^3\right)^2$

146. $\left(y^4\right)^3 \cdot \left(y^5\right)^2$

147. $\left(a^2\right)^6 \cdot \left(a^3\right)^8$

148. $\left(b^7\right)^5 \cdot \left(b^2\right)^6$

149. $(2m^6)^3$

150. $(3y^2)^4$

151. $(10x^2y)^3$

152. $(2mn^4)^5$

153. $(-2a^3b^2)^4$

154. $(-10u^2v^4)^3$

155. $\left(\frac{2}{3}x^2y\right)^3$

156. $\left(\frac{7}{9}pq^4\right)^2$

157. $(8a^3)^2(2a)^4$

158. $(5r^2)^3(3r)^2$

159. $(10p^4)^3(5p^6)^2$

160. $(4x^3)^3(2x^5)^4$

161. $\left(\frac{1}{2}x^2y^3\right)^4\left(4x^5y^3\right)^2$

162. $\left(\frac{1}{3}m^3n^2\right)^4\left(9m^8n^3\right)^2$

163. $(3m^2n)^2(2mn^5)^4$

164. $(2pq^4)^3(5p^6q)^2$

Everyday Math

165. Email Kate emails a flyer to ten of her friends and tells them to forward it to ten of their friends, who forward it to ten of their friends, and so on. The number of people who receive the email on the second round is 10^2, on the third round is 10^3, as shown in the table below. How many people will receive the email on the sixth round? Simplify the expression to show the number of people who receive the email.

Round	Number of people
1	10
2	10^2
3	10^3
…	…
6	?

166. Salary Jamal's boss gives him a 3% raise every year on his birthday. This means that each year, Jamal's salary is 1.03 times his last year's salary. If his original salary was $35,000, his salary after 1 year was $\$35{,}000(1.03)$, after 2 years was $\$35{,}000(1.03)^2$, after 3 years was $\$35{,}000(1.03)^3$, as shown in the table below. What will Jamal's salary be after 10 years? Simplify the expression, to show Jamal's salary in dollars.

Year	Salary
1	$\$35{,}000(1.03)$
2	$\$35{,}000(1.03)^2$
3	$\$35{,}000(1.03)^3$
…	…
10	?

167. Clearance A department store is clearing out merchandise in order to make room for new inventory. The plan is to mark down items by 30% each week. This means that each week the cost of an item is 70% of the previous week's cost. If the original cost of a sofa was $1,000, the cost for the first week would be $\$1,000(0.70)$ and the cost of the item during the second week would be $\$1,000(0.70)^2$. Complete the table shown below. What will be the cost of the sofa during the fifth week? Simplify the expression, to show the cost in dollars.

Week	Cost
1	$\$1,000(0.70)$
2	$\$1,000(0.70)^2$
3	
...	...
8	?

168. Depreciation Once a new car is driven away from the dealer, it begins to lose value. Each year, a car loses 10% of its value. This means that each year the value of a car is 90% of the previous year's value. If a new car was purchased for $20,000, the value at the end of the first year would be $\$20,000(0.90)$ and the value of the car after the end of the second year would be $\$20,000(0.90)^2$. Complete the table shown below. What will be the value of the car at the end of the eighth year? Simplify the expression, to show the value in dollars.

Week	Cost
1	$\$20,000(0.90)$
2	$\$20,000(0.90)^2$
3	
4	...
5	?

Writing Exercises

169. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

170. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

171. Jorge thinks $\left(\frac{1}{2}\right)^2$ is 1. What is wrong with his reasoning?

172. Explain why $x^3 \cdot x^5$ is x^8, and not x^{15}.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with exponents.			
simplify expressions using the Product Property for Exponents.			
simplify expressions using the Power Property for Exponents.			
simplify expressions using the Product to a Power Property.			
simplify expressions by applying several properties.			
multiply monomials.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

6.3 Multiply Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a trinomial by a binomial

Be Prepared!

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.
   If you missed this problem, review **Example 1.132**.
2. Combine like terms: $x^2+9x+7x+63$.
   If you missed this problem, review **Example 1.24**.

Multiply a Polynomial by a Monomial

We have used the Distributive Property to simplify expressions like $2(x-3)$. You multiplied both terms in the parentheses, $x \text{ and } 3$, by 2, to get $2x-6$. With this chapter's new vocabulary, you can say you were multiplying a binomial, $x-3$, by a monomial, 2.

Multiplying a binomial by a monomial is nothing new for you! Here's an example:

EXAMPLE 6.28

Multiply: $4(x+3)$.

Solution

	$4(x+3)$
Distribute.	$4 \cdot x + 4 \cdot 3$
Simplify.	$4x + 12$

TRY IT : : 6.55 Multiply: $5(x+7)$.

TRY IT : : 6.56 Multiply: $3(y+13)$.

EXAMPLE 6.29

Multiply: $y(y-2)$.

Solution

	$y(y-2)$
Distribute.	$y \cdot y - y \cdot 2$
Simplify.	$y^2 - 2y$

TRY IT : : 6.57 Multiply: $x(x-7)$.

TRY IT : : 6.58 Multiply: $d(d-11)$.

EXAMPLE 6.30

Multiply: $7x(2x+y)$.

Solution

	$7x(2x+y)$
Distribute.	$7x \cdot 2x + 7x \cdot y$
Simplify.	$14x^2 + 7xy$

TRY IT : : 6.59 Multiply: $5x(x+4y)$.

TRY IT : : 6.60 Multiply: $2p(6p+r)$.

EXAMPLE 6.31

Multiply: $-2y\left(4y^2+3y-5\right)$.

Solution

	$-2y(4y^2+3y-5)$
Distribute.	$-2y \cdot 4y^2 + (-2y) \cdot 3y - (-2y) \cdot 5$
Simplify.	$-8y^3 - 6y^2 + 10y$

TRY IT : : 6.61 Multiply: $-3y\left(5y^2+8y-7\right)$.

TRY IT : : 6.62 Multiply: $4x^2(2x^2-3x+5)$.

EXAMPLE 6.32

Multiply: $2x^3(x^2-8x+1)$.

Solution

	$2x^3(x^2 - 8x + 1)$
Distribute.	$2x^3 \cdot x^2 + (2x^3) \cdot (-8x) + (2x^3) \cdot 1$
Simplify.	$2x^5 - 16x^4 + 2x^3$

TRY IT : : 6.63 Multiply: $4x\left(3x^2 - 5x + 3\right)$.

TRY IT : : 6.64 Multiply: $-6a^3(3a^2 - 2a + 6)$.

EXAMPLE 6.33

Multiply: $(x + 3)p$.

Solution

The monomial is the second factor.	$(x + 3)p$
Distribute.	$x \cdot p + 3 \cdot p$
Simplify.	$xp + 3p$

TRY IT : : 6.65 Multiply: $(x + 8)p$.

TRY IT : : 6.66 Multiply: $(a + 4)p$.

Multiply a Binomial by a Binomial

Just like there are different ways to represent multiplication of numbers, there are several methods that can be used to multiply a binomial times a binomial. We will start by using the Distributive Property.

Multiply a Binomial by a Binomial Using the Distributive Property

Look at Example 6.33, where we multiplied a binomial by a monomial.

	$(x+3)p$
We distributed the p to get:	$xp+3p$
What if we have $(x+7)$ instead of p?	$(x+3)(x+7)$
Distribute $(x+7)$.	$x(x+7)+3(x+7)$
Distribute again.	$x^2+7x+3x+21$
Combine like terms.	$x^2+10x+21$

Notice that before combining like terms, you had four terms. You multiplied the two terms of the first binomial by the two terms of the second binomial—four multiplications.

EXAMPLE 6.34

Multiply: $(y+5)(y+8)$.

Solution

	$(y+5)(y+8)$
Distribute $(y+8)$.	$y(y+8)+5(y+8)$
Distribute again	$y^2+8y+5y+40$
Combine like terms.	$y^2+13y+40$

TRY IT : : 6.67 Multiply: $(x+8)(x+9)$.

TRY IT : : 6.68 Multiply: $(5x+9)(4x+3)$.

EXAMPLE 6.35

Multiply: $(2y+5)(3y+4)$.

Solution

	$(2y+5)(3y+4)$
Distribute $(3y+4)$.	$2y(3y+4)+5(3y+4)$
Distribute again	$6y^2+8y+15y+20$
Combine like terms.	$6y^2+23y+20$

TRY IT : : 6.69 Multiply: $(3b+5)(4b+6)$.

TRY IT : : 6.70 Multiply: $(a+10)(a+7)$.

EXAMPLE 6.36

Multiply: $(4y+3)(2y-5)$.

Solution

	$(4y+3)(2y-5)$
Distribute.	$4y(2y-5)+3(2y-5)$
Distribute again.	$8y^2-20y+6y-15$
Combine like terms.	$8y^2-14y-15$

TRY IT : : 6.71 Multiply: $(5y+2)(6y-3)$.

TRY IT : : 6.72 Multiply: $(3c+4)(5c-2)$.

EXAMPLE 6.37

Multiply: $(x+2)(x-y)$.

Solution

	$(x-2)(x-y)$
Distribute.	$x(x-y)-2(x-y)$
Distribute again.	$x^2-xy-2x+2y$
There are no like terms to combine.	

TRY IT : : 6.73 Multiply: $(a+7)(a-b)$.

TRY IT : : 6.74 Multiply: $(x+5)(x-y)$.

Multiply a Binomial by a Binomial Using the FOIL Method

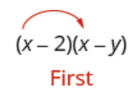
Remember that when you multiply a binomial by a binomial you get four terms. Sometimes you can combine like terms to get a trinomial, but sometimes, like in **Example 6.37**, there are no like terms to combine.

Let's look at the last example again and pay particular attention to how we got the four terms.

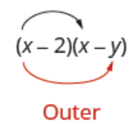
$$(x-2)(x-y)$$
$$x^2 - xy - 2x + 2y$$

Where did the first term, x^2, come from?

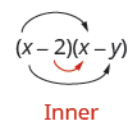
It is the product of x and x, the *first* terms in $(x-2)$ and $(x-y)$.



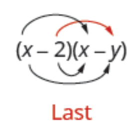
The next term, $-xy$, is the product of x and $-y$, the two *outer* terms.



The third term, $-2x$, is the product of -2 and x, the two *inner* terms.



And the last term, $+2y$, came from multiplying the two *last* terms, -2 and $-y$.



We abbreviate "First, Outer, Inner, Last" as FOIL. The letters stand for '**F**irst, **O**uter, **I**nner, **L**ast'. The word FOIL is easy to remember and ensures we find all four products.

$$(x-2)(x-y)$$
$$\underset{F}{x^2} - \underset{O}{xy} - \underset{I}{2x} + \underset{L}{2y}$$

Let's look at $(x+3)(x+7)$.

Distibutive Property	FOIL
$(x+3)(x+7)$	$(x+3)(x+7)$
$x(x+7)+3(x+7)$	
$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$	$\underset{F}{x^2} + \underset{O}{7x} + \underset{I}{3x} + \underset{L}{21}$
$x^2+10x+21$	$x^2+10x+21$

Notice how the terms in third line fit the FOIL pattern.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

EXAMPLE 6.38 HOW TO MULTIPLY A BINOMIAL BY A BINOMIAL USING THE FOIL METHOD

Multiply using the FOIL method: $(x+5)(x+9)$.

Solution

Step 1. Multiply the *First* terms.	$(x+5)(x+9)$ $(x+5)(x+9)$	$x^2 + \_ + \_ + \_$ F O I L
Step 2. Multiply the *Outer* terms.	$(x+5)(x+9)$	$x^2 + 9x + \_ + \_$ F O I L
Step 3. Multiply the *Inner* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + \_$ F O I L
Step 4. Multiply the *Last* terms.	$(x+5)(x+9)$	$x^2 + 9x + 5x + 45$ F O I L
Step 5. Combine like terms, when possible.		$x^2 + 14x + 45$

TRY IT : : 6.75 Multiply using the FOIL method: $(x+6)(x+8)$.

TRY IT : : 6.76 Multiply using the FOIL method: $(y+17)(y+3)$.

We summarize the steps of the FOIL method below. The FOIL method only applies to multiplying binomials, not other polynomials!

HOW TO : : MULTIPLY TWO BINOMIALS USING THE FOIL METHOD

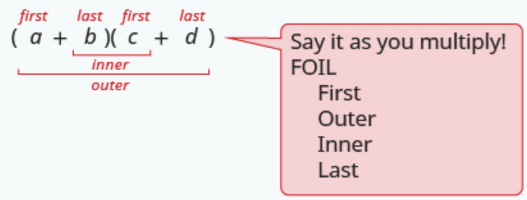
Step 1. Multiply the *First* terms.

Step 2. Multiply the *Outer* terms.

Step 3. Multiply the *Inner* terms.

Step 4. Multiply the *Last* terms.

Step 5. Combine like terms, when possible.



When you multiply by the FOIL method, drawing the lines will help your brain focus on the pattern and make it easier to apply.

EXAMPLE 6.39

Multiply: $(y-7)(y+4)$.

Solution

		$(y-7)(y+4)$
Multiply the *First* terms.	$(y-7)(y+4)$	$\underset{F}{y^2}+\underset{O}{\_}+\underset{I}{\_}+\underset{L}{\_}$
Multiply the *Outer* terms.	$(y-7)(y+4)$	$\underset{F}{y^2}+\underset{O}{4y}+\underset{I}{\_}+\underset{L}{\_}$
Multiply the *Inner* terms.	$(y-7)(y+4)$	$\underset{F}{y^2}+\underset{O}{4y}-\underset{I}{7y}+\underset{L}{\_}$
Multiply the *Last* terms.	$(y-7)(y+4)$	$\underset{F}{y^2}+\underset{O}{4y}-\underset{I}{7y}\underset{L}{-28}$
Combine like terms.		$y^2-3y-28$

TRY IT : : 6.77 Multiply: $(x-7)(x+5)$.

TRY IT : : 6.78 Multiply: $(b-3)(b+6)$.

EXAMPLE 6.40

Multiply: $(4x+3)(2x-5)$.

Solution

	$(4x-3)(2x-5)$
	$(4x+3)(2x-5)$
Multiply the *First* terms, $4x \cdot 2x$.	$\underset{F}{8x^2}+\underset{O}{\_}+\underset{I}{\_}+\underset{L}{\_}$
Multiply the *Outer* terms, $4x \cdot (-5)$.	$\underset{F}{8x^2}-\underset{O}{20x}+\underset{I}{\_}+\underset{L}{\_}$
Multiply the *Inner* terms, $3 \cdot 2x$.	$\underset{F}{8x^2}-\underset{O}{2x}+\underset{I}{6x}+\underset{L}{\_}$
Multiply the *Last* terms, $3 \cdot (-5)$.	$\underset{F}{8x^2}-\underset{O}{20x}+\underset{I}{6x}\underset{L}{-15}$
Combine like terms.	$8x^2-14x-15$

TRY IT : : 6.79 Multiply: $(3x+7)(5x-2)$.

TRY IT : : 6.80 Multiply: $(4y+5)(4y-10)$.

The final products in the last four examples were trinomials because we could combine the two middle terms. This is not always the case.

EXAMPLE 6.41

Multiply: $(3x-y)(2x-5)$.

Solution

	$(3x - y)(2x - 5)$
	$(3x - y)(2x - 5)$
Multiply the *First*.	$\underset{F}{6x^2} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Outer*.	$\underset{F}{6x^2} \underset{O}{- 15x} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Inner*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} + \underset{L}{\_}$
Multiply the *Last*.	$\underset{F}{6x^2} \underset{O}{- 15x} \underset{I}{- 2xy} \underset{L}{+ 5y}$
Combine like terms—there are none.	$6x^2 - 15x - 2xy + 5y$

TRY IT : : 6.81 Multiply: $(10c - d)(c - 6)$.

TRY IT : : 6.82 Multiply: $(7x - y)(2x - 5)$.

Be careful of the exponents in the next example.

EXAMPLE 6.42

Multiply: $\left(n^2 + 4\right)(n - 1)$.

Solution

	$(n^2 + 4)(n - 1)$
	$(n^2 + 4)(n - 1)$
Multiply the *First*.	$\underset{F}{n^3} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Outer*.	$\underset{F}{n^3} \underset{O}{- n^2} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Inner*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} + \underset{L}{\_}$
Multiply the *Last*.	$\underset{F}{n^3} \underset{O}{- n^2} \underset{I}{+ 4n} \underset{L}{- 4}$
Combine like terms—there are none.	$n^3 - n^2 + 4n - 4$

TRY IT : : 6.83 Multiply: $\left(x^2 + 6\right)(x - 8)$.

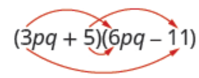
TRY IT : : 6.84 Multiply: $\left(y^2 + 7\right)(y - 9)$.

EXAMPLE 6.43

Multiply: $(3pq + 5)(6pq - 11)$.

Solution

	$(3pq + 5)(6pq - 11)$
Multiply the *First*.	$\underset{F}{18p^2q^2} + \underset{O}{\_} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Outer*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} + \underset{I}{\_} + \underset{L}{\_}$
Multiply the *Inner*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} + \underset{L}{\_}$
Multiply the *Last*.	$\underset{F}{18p^2q^2} \underset{O}{- 33pq} \underset{I}{+ 30pq} \underset{L}{- 55}$
Combine like terms—there are none.	$18p^2q^2 - 3pq - 55$



TRY IT : : 6.85 Multiply: $(2ab + 5)(4ab - 4)$.

TRY IT : : 6.86 Multiply: $(2xy + 3)(4xy - 5)$.

Multiply a Binomial by a Binomial Using the Vertical Method

The FOIL method is usually the quickest method for multiplying two binomials, but it *only* works for binomials. You can use the Distributive Property to find the product of any two polynomials. Another method that works for all polynomials is the Vertical Method. It is very much like the method you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

23	
×46	
138 partial product	Start by multiplying 23 by 6 to get 138.
92 partial product	Next, multiply 23 by 4, lining up the partial product in the correct columns.
1058 product	Last you add the partial products.

Now we'll apply this same method to multiply two binomials.

EXAMPLE 6.44

Multiply using the Vertical Method: $(3y - 1)(2y - 6)$.

Solution

It does not matter which binomial goes on the top.

Multiply $3y - 1$ by -6.	$3y - 1$	
Multiply $3y - 1$ by $2y$.	$\times\ 2y - 6$	
	$-18y + 6$	partial product
	$6y^2 - 2y$	partial product
Add like terms.	$6y^2 - 20y + 6$	product

Notice the partial products are the same as the terms in the FOIL method.

$$(3y-1)(2y-6)$$
$$6y^2-2y-18y+6$$
$$6y^2-20y+6$$

$$\begin{array}{r} 3y-1 \\ \times\ 2y-6 \\ \hline -18y+6 \\ 6y^2-2y \quad\quad \\ \hline 6y^2-20x+6 \end{array}$$

TRY IT : : 6.87 Multiply using the Vertical Method: $(5m-7)(3m-6)$.

TRY IT : : 6.88 Multiply using the Vertical Method: $(6b-5)(7b-3)$.

We have now used three methods for multiplying binomials. Be sure to practice each method, and try to decide which one you prefer. The methods are listed here all together, to help you remember them.

Multiplying Two Binomials

To multiply binomials, use the:

- Distributive Property
- FOIL Method
- Vertical Method

Remember, FOIL only works when multiplying two binomials.

Multiply a Trinomial by a Binomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a trinomial by a binomial. Remember, FOIL will not work in this case, but we can use either the Distributive Property or the Vertical Method. We first look at an example using the Distributive Property.

EXAMPLE 6.45

Multiply using the Distributive Property: $(b+3)(2b^2-5b+8)$.

Solution

	$(b+3)(2b^2-5b+8)$
Distribute.	$b(2b^2-5b+8)+3(2b^2-5b+8)$
Multiply.	$2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$2b^3+b^2-7b+24$

TRY IT : : 6.89 Multiply using the Distributive Property: $(y-3)(y^2-5y+2)$.

TRY IT : : 6.90 Multiply using the Distributive Property: $(x+4)(2x^2-3x+5)$.

Now let's do this same multiplication using the Vertical Method.

EXAMPLE 6.46

Multiply using the Vertical Method: $(b+3)(2b^2-5b+8)$.

Solution

It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

Multiply $(2b^2 - 5b + 8)$ by 3.	$\begin{array}{r} 2b^2 - 5b + 8 \\ \times \quad b + 3 \\ \hline 6b^2 - 15b + 24 \end{array}$
	$2b^3 - 5b^2 + 8b$
Multiply $(2b^2 - 5b + 8)$ by b.	$2b^3 + b^2 - 7b + 24$
Add like terms.	

TRY IT : : 6.91 Multiply using the Vertical Method: $(y - 3)(y^2 - 5y + 2)$.

TRY IT : : 6.92 Multiply using the Vertical Method: $(x + 4)(2x^2 - 3x + 5)$.

We have now seen two methods you can use to multiply a trinomial by a binomial. After you practice each method, you'll probably find you prefer one way over the other. We list both methods are listed here, for easy reference.

Multiplying a Trinomial by a Binomial

To multiply a trinomial by a binomial, use the:

- Distributive Property
- Vertical Method

MEDIA : :

Access these online resources for additional instruction and practice with multiplying polynomials:

- Multiplying Exponents 1 (https://openstax.org/l/25MultiplyExp1)
- Multiplying Exponents 2 (https://openstax.org/l/25MultiplyExp2)
- Multiplying Exponents 3 (https://openstax.org/l/25MultiplyExp3)

6.3 EXERCISES

Practice Makes Perfect

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

173. $4(w+10)$

174. $6(b+8)$

175. $-3(a+7)$

176. $-5(p+9)$

177. $2(x-7)$

178. $7(y-4)$

179. $-3(k-4)$

180. $-8(j-5)$

181. $q(q+5)$

182. $k(k+7)$

183. $-b(b+9)$

184. $-y(y+3)$

185. $-x(x-10)$

186. $-p(p-15)$

187. $6r(4r+s)$

188. $5c(9c+d)$

189. $12x(x-10)$

190. $9m(m-11)$

191. $-9a(3a+5)$

192. $-4p(2p+7)$

193. $3(p^2+10p+25)$

194. $6(y^2+8y+16)$

195. $-8x(x^2+2x-15)$

196. $-5t(t^2+3t-18)$

197. $5q^3\left(q^3-2q+6\right)$

198. $4x^3\left(x^4-3x+7\right)$

199. $-8y(y^2+2y-15)$

200. $-5m(m^2+3m-18)$

201. $5q^3(q^2-2q+6)$

202. $9r^3(r^2-3r+5)$

203. $-4z^2(3z^2+12z-1)$

204. $-3x^2(7x^2+10x-1)$

205. $(2m-9)m$

206. $(8j-1)j$

207. $(w-6)\cdot 8$

208. $(k-4)\cdot 5$

209. $4(x+10)$

210. $6(y+8)$

211. $15(r-24)$

212. $12(v-30)$

213. $-3(m+11)$

214. $-4(p+15)$

215. $-8(z-5)$

216. $-3(x-9)$

217. $u(u+5)$

218. $q(q+7)$

219. $n(n^2-3n)$

220. $s(s^2-6s)$

221. $6x(4x+y)$

222. $5a(9a+b)$

223. $5p(11p-5q)$

224. $12u(3u-4v)$

225. $3(v^2+10v+25)$

226. $6(x^2+8x+16)$

227. $2n(4n^2-4n+1)$

228. $3r(2r^2-6r+2)$

229. $-8y(y^2+2y-15)$

230. $-5m(m^2+3m-18)$

231. $5q^3(q^2-2q+6)$

232. $9r^3(r^2-3r+5)$

233. $-4z^2(3z^2 + 12z - 1)$

234. $-3x^2(7x^2 + 10x - 1)$

235. $(2y - 9)y$

236. $(8b - 1)b$

Multiply a Binomial by a Binomial

In the following exercises, multiply the following binomials using: ⓐ the Distributive Property ⓑ the FOIL method ⓒ the Vertical Method.

237. $(w + 5)(w + 7)$

238. $(y + 9)(y + 3)$

239. $(p + 11)(p - 4)$

240. $(q + 4)(q - 8)$

In the following exercises, multiply the binomials. Use any method.

241. $(x + 8)(x + 3)$

242. $(y + 7)(y + 4)$

243. $(y - 6)(y - 2)$

244. $(x - 7)(x - 2)$

245. $(w - 4)(w + 7)$

246. $(q - 5)(q + 8)$

247. $(p + 12)(p - 5)$

248. $(m + 11)(m - 4)$

249. $(6p + 5)(p + 1)$

250. $(7m + 1)(m + 3)$

251. $(2t - 9)(10t + 1)$

252. $(3r - 8)(11r + 1)$

253. $(5x - y)(3x - 6)$

254. $(10a - b)(3a - 4)$

255. $(a + b)(2a + 3b)$

256. $(r + s)(3r + 2s)$

257. $(4z - y)(z - 6)$

258. $(5x - y)(x - 4)$

259. $(x^2 + 3)(x + 2)$

260. $(y^2 - 4)(y + 3)$

261. $(x^2 + 8)(x^2 - 5)$

262. $(y^2 - 7)(y^2 - 4)$

263. $(5ab - 1)(2ab + 3)$

264. $(2xy + 3)(3xy + 2)$

265. $(6pq - 3)(4pq - 5)$

266. $(3rs - 7)(3rs - 4)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property ⓑ the Vertical Method.

267. $(x + 5)(x^2 + 4x + 3)$

268. $(u + 4)(u^2 + 3u + 2)$

269. $(y + 8)(4y^2 + y - 7)$

270. $(a + 10)(3a^2 + a - 5)$

In the following exercises, multiply. Use either method.

271. $(w - 7)(w^2 - 9w + 10)$

272. $(p - 4)(p^2 - 6p + 9)$

273. $(3q + 1)(q^2 - 4q - 5)$

274. $(6r + 1)(r^2 - 7r - 9)$

Mixed Practice

275. $(10y - 6) + (4y - 7)$

276. $(15p - 4) + (3p - 5)$

277. $\left(x^2 - 4x - 34\right) - \left(x^2 + 7x - 6\right)$

278. $\left(j^2 - 8j - 27\right) - \left(j^2 + 2j - 12\right)$

279. $5q(3q^2 - 6q + 11)$

280. $8t(2t^2 - 5t + 6)$

281. $(s - 7)(s + 9)$

282. $(x - 5)(x + 13)$

283. $\left(y^2 - 2y\right)(y + 1)$

284. $\left(a^2 - 3a\right)(4a + 5)$

285. $(3n - 4)\left(n^2 + n - 7\right)$

286. $(6k - 1)\left(k^2 + 2k - 4\right)$

287. $(7p + 10)(7p - 10)$

288. $(3y + 8)(3y - 8)$

289. $(4m^2 - 3m - 7)m^2$

290. $(15c^2 - 4c + 5)c^4$

291. $(5a + 7b)(5a + 7b)$

292. $(3x - 11y)(3x - 11y)$

293. $(4y + 12z)(4y - 12z)$

Everyday Math

294. Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 13 times 15. Think of 13 as $10 + 3$ and 15 as $10 + 5$.

ⓐ Multiply $(10 + 3)(10 + 5)$ by the FOIL method.

ⓑ Multiply $13 \cdot 15$ without using a calculator.

ⓒ Which way is easier for you? Why?

295. Mental math You can use binomial multiplication to multiply numbers without a calculator. Say you need to multiply 18 times 17. Think of 18 as $20 - 2$ and 17 as $20 - 3$.

ⓐ Multiply $(20 - 2)(20 - 3)$ by the FOIL method.

ⓑ Multiply $18 \cdot 17$ without using a calculator.

ⓒ Which way is easier for you? Why?

Writing Exercises

296. Which method do you prefer to use when multiplying two binomials: the Distributive Property, the FOIL method, or the Vertical Method? Why?

297. Which method do you prefer to use when multiplying a trinomial by a binomial: the Distributive Property or the Vertical Method? Why?

298. Multiply the following:

$(x + 2)(x - 2)$

$(y + 7)(y - 7)$

$(w + 5)(w - 5)$

Explain the pattern that you see in your answers.

299. Multiply the following:

$(m - 3)(m + 3)$

$(n - 10)(n + 10)$

$(p - 8)(p + 8)$

Explain the pattern that you see in your answers.

300. Multiply the following:

$(p + 3)(p + 3)$

$(q + 6)(q + 6)$

$(r + 1)(r + 1)$

Explain the pattern that you see in your answers.

301. Multiply the following:

$(x - 4)(x - 4)$

$(y - 1)(y - 1)$

$(z - 7)(z - 7)$

Explain the pattern that you see in your answers.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
multiply a polynomial by a monomial.			
multiply a binomial by a binomial.			
multiply a trinomial by a binomial.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

6.4 Special Products

Learning Objectives

By the end of this section, you will be able to:

- Square a binomial using the Binomial Squares Pattern
- Multiply conjugates using the Product of Conjugates Pattern
- Recognize and use the appropriate special product pattern

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ 9^2 ⓑ $(-9)^2$ ⓒ -9^2.
   If you missed this problem, review **Example 1.50**.

Square a Binomial Using the Binomial Squares Pattern

Mathematicians like to look for patterns that will make their work easier. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and using the methods of the last section, there is less work to do if you learn to use a pattern.

Let's start by looking at $(x+9)^2$.

What does this mean?	$(x+9)^2$
It means to multiply $(x+9)$ by itself.	$(x+9)(x+9)$
Then, using FOIL, we get:	$x^2+9x+9x+81$
Combining like terms gives:	$x^2+18x+81$

Here's another one:	$(y-7)^2$
Multiply $(y-7)$ by itself.	$(y-7)(y-7)$
Using FOIL, we get:	$y^2-7y-7y+49$
And combining like terms:	$y^2-14y+49$

And one more:	$(2x+3)^2$
Multiply.	$(2x+3)(2x+3)$
Use FOIL:	$4x^2+6x+6x+9$
Combine like terms.	$4x^2+12x+9$

Look at these results. Do you see any patterns?

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_$$

Now look at the ***first term*** in each result. Where did it come from?

$(x+9)^2$	$(y-7)^2$	$(2x+3)^2$
$(x+9)(x+9)$	$(y-7)(y-7)$	$(2x+3)(2x+3)$
$x^2+9x+9x+81$	$y^2-7y-7y+49$	$4x^2+6x+6x+9$
$x^2+18x+81$	$y^2-14y+49$	$4x^2+12x+9$

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2 = a^2 + \underline{\quad\quad} + \underline{\quad\quad}$$

To get the ***first term*** of the product, ***square the first term***.

Where did the ***last term*** come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2 = \underline{\quad\quad} + \underline{\quad\quad} + b^2$$

*To get the **last term** of the product, **square the last term**.*

Finally, look at the ***middle term***. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2 = \underline{\quad\quad} + 2ab + \underline{\quad\quad}$$
$$(a-b)^2 = \underline{\quad\quad} - 2ab + \underline{\quad\quad}$$

*To get the **middle term** of the product, **multiply the terms and double their product**.*

Putting it all together:

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}$$

To square a binomial:

- square the first term
- square the last term
- double their product

A number example helps verify the pattern.

	$(10+4)^2$
Square the fir t term.	$10^2 + \underline{\quad} + \underline{\quad}$
Square the last term.	$10^2 + \underline{\quad} + 4^2$
Double their product.	$10^2 + 2\cdot 10\cdot 4 + 4^2$
Simplify.	$100 + 80 + 16$
Simplify.	196

To multiply $(10+4)^2$ usually you'd follow the Order of Operations.

$$(10+4)^2$$
$$(14)^2$$
$$196$$

The pattern works!

EXAMPLE 6.47

Multiply: $(x+5)^2$.

Solution

	$\overset{a+b}{(x+5)}^2$
Square the first term.	$\overset{a^2+\;2ab\;+\;b^2}{x^2+\_\_\_+\_\_\_}$
Square the last term.	$\overset{a^2+\;2ab\;+\;b^2}{x^2+\_\_\_+5^2}$
Double the product.	$\overset{a^2+2\cdot a\cdot b+b^2}{x^2+2\cdot x\cdot 5+5^2}$
Simplify.	$x^2+10x+25$

TRY IT : : 6.93 Multiply: $(x+9)^2$.

TRY IT : : 6.94 Multiply: $(y+11)^2$.

EXAMPLE 6.48

Multiply: $(y-3)^2$.

Solution

	$\overset{a-b}{(y-3)}^2$
Square the first term.	$\overset{a^2-\;2ab\;+\;b^2}{y^2-\_\_\_+\_\_\_}$
Square the last term.	$\overset{a^2-\;2ab\;+\;b^2}{y^2-\_\_\_+3^2}$
Double the product.	$\overset{a^2-2\cdot a\cdot b+b^2}{y^2-2\cdot y\cdot 3+3^2}$
Simplify.	y^2-6y+9

TRY IT : : 6.95 Multiply: $(x-9)^2$.

TRY IT : : 6.96 Multiply: $(p-13)^2$.

EXAMPLE 6.49

Multiply: $(4x+6)^2$.

Solution

	$\overset{a\ +\ b}{(4x+6)}^2$
Use the pattern.	$\overset{a^2\ +\ 2\cdot a\cdot b\ +\ b^2}{(4x)^2+2\cdot 4x\cdot 6+6^2}$
Simplify.	$16x^2+48x+36$

TRY IT : : 6.97 Multiply: $(6x+3)^2$.

TRY IT : : 6.98 Multiply: $(4x+9)^2$.

EXAMPLE 6.50

Multiply: $(2x-3y)^2$.

Solution

	$\overset{a\ -\ b}{(2x-3y)}^2$
Use the pattern.	$\overset{a^2\ -\ 2\cdot a\cdot b\ +\ b^2}{(2x)^2-2\cdot 2x\cdot 3y+(3y)^2}$
Simplify.	$4x^2-12xy+9y^2$

TRY IT : : 6.99 Multiply: $(2c-d)^2$.

TRY IT : : 6.100 Multiply: $(4x-5y)^2$.

EXAMPLE 6.51

Multiply: $\left(4u^3+1\right)^2$.

Solution

	$\overset{a\ +\ b}{(4u^3+1)}^2$
Use the pattern.	$\overset{a^2\ +\ 2\cdot a\cdot b\ +\ b^2}{(4u^3)^2+2\cdot 4u^3\cdot 1+(1)^2}$
Simplify.	$16u^6+8u^3+1$

TRY IT : : 6.101 Multiply: $\left(2x^2+1\right)^2$.

TRY IT : : 6.102 Multiply: $\left(3y^3+2\right)^2$.

Multiply Conjugates Using the Product of Conjugates Pattern

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

What do you notice about these pairs of binomials?

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Look at the first term of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the first terms are the same in each pair.

Look at the last terms of each binomial in each pair.

$$(x-9)(x+9) \qquad (y-8)(y+8) \qquad (2x-5)(2x+5)$$

Notice the last terms are the same in each pair.

Notice how each pair has one sum and one difference.

$(x-9)$ Difference, $(x+9)$ Sum $(y-8)$ Difference, $(y+8)$ Sum $(2x-5)$ Difference, $(2x+5)$ Sum

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference has a special name. It is called a *conjugate pair* and is of the form $(a-b),\ (a+b)$.

Conjugate Pair

A **conjugate pair** is two binomials of the form

$$(a-b),\ (a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier.

Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$(x-9)(x+9)$	$(y-8)(y+8)$	$(2x-5)(2x+5)$
$x^2+9x-9x-81$	$y^2+8y-8y-64$	$4x^2+10x-10x-25$
x^2-81	y^2-64	$4x^2-25$

$(x+9)(x-9)$	$(y-8)(y+8)$	$(2x-5)(2x+5)$
$x^2-9x+9x-81$	$y^2+8y-8y-64$	$4x^2+10x-10x-25$
x^2-81	y^2-64	$4x^2-25$

Each **first term** is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\underline{\quad\quad}$$

To get the **fir t term, square the fir t term**.

The **last term** came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the **last term, square the last term**.

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a difference of squares.

This leads to the pattern:

Product of Conjugates Pattern

If a and b are real numbers,

$$(a-b)(a+b) = a^2 - b^2$$

$$\underbrace{(a-b)(a+b)}_{\text{conjugates}} = \underbrace{a^2}_{\text{squares}} \overset{\text{difference}}{-} \underbrace{b^2}_{\text{squares}}$$

The product is called a difference of squares.

To multiply conjugates, square the first term, square the last term, and write the product as a difference of squares.

Let's test this pattern with a numerical example.

	$(10-2)(10+2)$
It is the product of conjugates, so the result will be the diffe ence of two squares.	$\_\_\_\_ - \_\_\_\_$
Square the fir t term.	$10^2 - \_\_\_\_$
Square the last term.	$10^2 - 2^2$
Simplify.	$100 - 4$
Simplify.	96
What do you get using the Order of Operations?	
	$(10-2)(10+2)$
	$(8)(12)$
	96

Notice, the result is the same!

EXAMPLE 6.52

Multiply: $(x-8)(x+8)$.

Solution

First, recognize this as a product of conjugates. The binomials have the same first terms, and the same last terms, and one binomial is a sum and the other is a difference.

It fits the pattern.	$\overset{a\ -\ b}{(x-8)}\overset{a\ +\ b}{(x+8)}$
Square the first term, x.	$\overset{a^2\ -\ b^2}{x^2 - \_\_}$
Square the last term, 8.	$\overset{a^2\ -\ b^2}{x^2 - 8^2}$
The product is a difference of squares.	$\overset{a^2\ -\ b^2}{x^2 - 64}$

TRY IT : : 6.103 Multiply: $(x-5)(x+5)$.

TRY IT : : 6.104 Multiply: $(w-3)(w+3)$.

EXAMPLE 6.53

Multiply: $(2x + 5)(2x - 5)$.

Solution

Are the binomials conjugates?

It is the product of conjugates.	$\overset{a\ +\ b}{(2x + 5)}\overset{a\ -\ b}{(2x - 5)}$
Square the first term, $2x$.	$\overset{a^2\ -\ b^2}{(2x)^2 - \_\_}$
Square the last term, 5.	$\overset{a^2\ -\ b^2}{(2x)^2 - 5^2}$
Simplify. The product is a difference of squares.	$\overset{a^2\ -\ b^2}{4x^2 - 25}$

TRY IT : : 6.105 Multiply: $(6x + 5)(6x - 5)$.

TRY IT : : 6.106 Multiply: $(2x + 7)(2x - 7)$.

The binomials in the next example may look backwards – the variable is in the second term. But the two binomials are still conjugates, so we use the same pattern to multiply them.

EXAMPLE 6.54

Find the product: $(3 + 5x)(3 - 5x)$.

Solution

It is the product of conjugates.	$\overset{a\ -\ b}{(3 + 5x)}\overset{a\ +\ b}{(3 - 5x)}$
Use the pattern.	$\overset{a^2\ -\ b^2}{3^2 - (5x)^2}$
Simplify.	$9 - 25x^2$

TRY IT : : 6.107 Multiply: $(7 + 4x)(7 - 4x)$.

TRY IT : : 6.108 Multiply: $(9 - 2y)(9 + 2y)$.

Now we'll multiply conjugates that have two variables.

EXAMPLE 6.55

Find the product: $(5m - 9n)(5m + 9n)$.

Solution

This fits the pattern.	$\overset{a}{(5m} \overset{-}{-} \overset{b}{9n)}\overset{a}{(5m} \overset{+}{+} \overset{b}{9n)}$
Use the pattern.	$\overset{a^2}{(5m)^2} \overset{-}{-} \overset{b^2}{(9n)^2}$
Simplify.	$25m^2 - 81n^2$

TRY IT : : 6.109 Find the product: $(4p - 7q)(4p + 7q)$.

TRY IT : : 6.110 Find the product: $(3x - y)(3x + y)$.

EXAMPLE 6.56

Find the product: $(cd - 8)(cd + 8)$.

Solution

This fits the pattern.	$\overset{a}{(cd} \overset{-}{-} \overset{b}{8)}\ \overset{a}{(cd} \overset{+}{+} \overset{b}{8)}$
Use the pattern.	$\overset{a^2}{(cd)^2} \overset{-}{-} \overset{b^2}{(8)^2}$
Simplify.	$c^2d^2 - 64$

TRY IT : : 6.111 Find the product: $(xy - 6)(xy + 6)$.

TRY IT : : 6.112 Find the product: $(ab - 9)(ab + 9)$.

EXAMPLE 6.57

Find the product: $\left(6u^2 - 11v^5\right)\left(6u^2 + 11v^5\right)$.

Solution

This fits the pattern.	$\overset{a}{(6u^2} \overset{-}{-} \overset{b}{11v^5)}\ \overset{a}{(6u^2} \overset{+}{+} \overset{b}{11v^5)}$
Use the pattern.	$\overset{a^2}{(6u^2)^2} \overset{-}{-} \overset{b^2}{(11v^5)^2}$
Simplify.	$36u^4 - 121v^{10}$

TRY IT : : 6.113 Find the product: $\left(3x^2 - 4y^3\right)\left(3x^2 + 4y^3\right)$.

TRY IT : : 6.114 Find the product: $\left(2m^2 - 5n^3\right)\left(2m^2 + 5n^3\right)$.

Recognize and Use the Appropriate Special Product Pattern

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

Comparing the Special Product Patterns

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)(a+b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	
- Squaring a binomial	- Multiplying conjugates
- Product is a **trinomial**	- Product is a **binomial**
- Inner and outer terms with FOIL are **the same.**	- Inner and outer terms with FOIL are **opposites.**
- Middle term is **double the product** of the terms.	- There is **no** middle term.

EXAMPLE 6.58

Choose the appropriate pattern and use it to find the product:

ⓐ $(2x-3)(2x+3)$ ⓑ $(5x-8)^2$ ⓒ $(6m+7)^2$ ⓓ $(5x-6)(6x+5)$

Solution

ⓐ $(2x-3)(2x+3)$ These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

This fits the pattern.	$\overset{a}{(2x} \overset{-\ b}{- 3)}\overset{a}{(2x} \overset{+\ b}{+ 3)}$
Use the pattern.	$\overset{a^2}{(2x)^2} \overset{-\ b^2}{- 3^2}$
Simplify.	$4x^2 - 9$

ⓑ $(8x-5)^2$ We are asked to square a binomial. It fits the **binomial squares** pattern.

	$\overset{a\ -\ b}{(8x-5)^2}$
Use the pattern.	$\overset{a^2}{(8x)^2} \overset{-}{-} \overset{2ab}{2 \cdot 8x \cdot 5} \overset{+\ b^2}{+ 5^2}$
Simplify.	$64x^2 - 80x + 25$

ⓒ $(6m+7)^2$ Again, we will square a binomial so we use the **binomial squares** pattern.

	$\overset{a\ +\ b}{(6m+7)^2}$
Use the pattern.	$\overset{a^2}{(6m)^2} \overset{+}{+} \overset{2ab}{2 \cdot 6m \cdot 7} \overset{+\ b^2}{+ 7^2}$
Simplify.	$36m^2 + 84m + 49$

ⓓ $(5x-6)(6x+5)$ This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$30x^2 + 25x - 36x - 30$
Simplify.	$30x^2 - 11x - 30$

TRY IT : : 6.115 Choose the appropriate pattern and use it to find the product:

ⓐ $(9b - 2)(2b + 9)$ ⓑ $(9p - 4)^2$ ⓒ $(7y + 1)^2$ ⓓ $(4r - 3)(4r + 3)$

TRY IT : : 6.116 Choose the appropriate pattern and use it to find the product:

ⓐ $(6x + 7)^2$ ⓑ $(3x - 4)(3x + 4)$ ⓒ $(2x - 5)(5x - 2)$ ⓓ $(6n - 1)^2$

MEDIA : :

Access these online resources for additional instruction and practice with special products:

- **Special Products (https://openstax.org/l/25Specialprod)**

6.4 EXERCISES

Practice Makes Perfect

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

302. $(w+4)^2$

303. $(q+12)^2$

304. $\left(y+\frac{1}{4}\right)^2$

305. $\left(x+\frac{2}{3}\right)^2$

306. $(b-7)^2$

307. $(y-6)^2$

308. $(m-15)^2$

309. $(p-13)^2$

310. $(3d+1)^2$

311. $(4a+10)^2$

312. $\left(2q+\frac{1}{3}\right)^2$

313. $\left(3z+\frac{1}{5}\right)^2$

314. $(3x-y)^2$

315. $(2y-3z)^2$

316. $\left(\frac{1}{5}x-\frac{1}{7}y\right)^2$

317. $\left(\frac{1}{8}x-\frac{1}{9}y\right)^2$

318. $(3x^2+2)^2$

319. $(5u^2+9)^2$

320. $(4y^3-2)^2$

321. $(8p^3-3)^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

322. $(m-7)(m+7)$

323. $(c-5)(c+5)$

324. $\left(x+\frac{3}{4}\right)\left(x-\frac{3}{4}\right)$

325. $\left(b+\frac{6}{7}\right)\left(b-\frac{6}{7}\right)$

326. $(5k+6)(5k-6)$

327. $(8j+4)(8j-4)$

328. $(11k+4)(11k-4)$

329. $(9c+5)(9c-5)$

330. $(11-b)(11+b)$

331. $(13-q)(13+q)$

332. $(5-3x)(5+3x)$

333. $(4-6y)(4+6y)$

334. $(9c-2d)(9c+2d)$

335. $(7w+10x)(7w-10x)$

336. $\left(m+\frac{2}{3}n\right)\left(m-\frac{2}{3}n\right)$

337. $\left(p+\frac{4}{5}q\right)\left(p-\frac{4}{5}q\right)$

338. $(ab-4)(ab+4)$

339. $(xy-9)(xy+9)$

340. $(uv-\frac{3}{5})(uv+\frac{3}{5})$

341. $(rs-\frac{2}{7})(rs+\frac{2}{7})$

342. $(2x^2-3y^4)(2x^2+3y^4)$

343. $(6m^3-4n^5)(6m^3+4n^5)$

344. $(12p^3-11q^2)(12p^3+11q^2)$

345. $(15m^2-8n^4)(15m^2+8n^4)$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

346.

ⓐ $(p-3)(p+3)$

ⓑ $(t-9)^2$

ⓒ $(m+n)^2$

ⓓ $(2x+y)(x-2y)$

347.

ⓐ $(2r+12)^2$

ⓑ $(3p+8)(3p-8)$

ⓒ $(7a+b)(a-7b)$

ⓓ $(k-6)^2$

348.

ⓐ $(a^5-7b)^2$

ⓑ $(x^2+8y)(8x-y^2)$

ⓒ $(r^6+s^6)(r^6-s^6)$

ⓓ $(y^4+2z)^2$

349.

ⓐ $(x^5+y^5)(x^5-y^5)$

ⓑ $(m^3-8n)^2$

ⓒ $(9p+8q)^2$

ⓓ $(r^2-s^3)(r^3+s^2)$

Everyday Math

350. Mental math You can use the product of conjugates pattern to multiply numbers without a calculator. Say you need to multiply 47 times 53. Think of 47 as $50-3$ and 53 as $50+3$.

ⓐ Multiply $(50-3)(50+3)$ by using the product of conjugates pattern, $(a-b)(a+b)=a^2-b^2$.

ⓑ Multiply $47 \cdot 53$ without using a calculator.

ⓒ Which way is easier for you? Why?

351. Mental math You can use the binomial squares pattern to multiply numbers without a calculator. Say you need to square 65. Think of 65 as $60+5$.

ⓐ Multiply $(60+5)^2$ by using the binomial squares pattern, $(a+b)^2=a^2+2ab+b^2$.

ⓑ Square 65 without using a calculator.

ⓒ Which way is easier for you? Why?

Writing Exercises

352. How do you decide which pattern to use?

353. Why does $(a+b)^2$ result in a trinomial, but $(a-b)(a+b)$ result in a binomial?

354. Marta did the following work on her homework paper:

$$(3-y)^2$$
$$3^2-y^2$$
$$9-y^2$$

Explain what is wrong with Marta's work.

355. Use the order of operations to show that $(3+5)^2$ is 64, and then use that numerical example to explain why $(a+b)^2 \neq a^2+b^2$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
square a binomial using the binomial squares pattern.			
multiply conjugates using the product of conjugates pattern.			
recognize and use the appropriate special product pattern.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

6.5 Divide Monomials

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the Quotient Property for Exponents
- Simplify expressions with zero exponents
- Simplify expressions using the quotient to a Power Property
- Simplify expressions by applying several properties
- Divide monomials

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{8}{24}$.

   If you missed this problem, review Example 1.65.

2. Simplify: $\left(2m^3\right)^5$.

   If you missed this problem, review Example 6.23.

3. Simplify: $\frac{12x}{12y}$.

   If you missed this problem, review Example 1.67.

Simplify Expressions Using the Quotient Property for Exponents

Earlier in this chapter, we developed the properties of exponents for multiplication. We summarize these properties below.

Summary of Exponent Properties for Multiplication

If a and b are real numbers, and m and n are whole numbers, then

$$\begin{array}{lrcl} \textbf{Product Property} & a^m \cdot a^n & = & a^{m+n} \\ \textbf{Power Property} & (a^m)^n & = & a^{m \cdot n} \\ \textbf{Product to a Power} & (ab)^m & = & a^m b^m \end{array}$$

Now we will look at the exponent properties for division. A quick memory refresher may help before we get started. You have learned to simplify fractions by dividing out common factors from the numerator and denominator using the Equivalent Fractions Property. This property will also help you work with algebraic fractions—which are also quotients.

Equivalent Fractions Property

If $a, b,$ and c are whole numbers where $b \neq 0, c \neq 0$,

$$\text{then} \quad \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	x^3		$\frac{1}{x}$

Notice, in each case the bases were the same and we subtracted exponents.

When the larger exponent was in the numerator, we were left with factors in the numerator.

When the larger exponent was in the denominator, we were left with factors in the denominator—notice the numerator of 1.

We write:

$$\begin{array}{cc} \frac{x^5}{x^2} & \frac{x^2}{x^3} \\ x^{5-2} & \frac{1}{x^{3-2}} \\ x^3 & \frac{1}{x} \end{array}$$

This leads to the *Quotient Property for Exponents*.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then

$$\frac{a^m}{a^n} = a^{m-n},\ m > n \text{ and } \frac{a^m}{a^n} = \frac{1}{a^{n-m}},\ n > m$$

A couple of examples with numbers may help to verify this property.

$$\begin{array}{rcl} \frac{3^4}{3^2} & = & 3^{4-2} \\ \frac{81}{9} & = & 3^2 \\ 9 & = & 9 \checkmark \end{array} \qquad \begin{array}{rcl} \frac{5^2}{5^3} & = & \frac{1}{5^{3-2}} \\ \frac{25}{125} & = & \frac{1}{5^1} \\ \frac{1}{5} & = & \frac{1}{5} \checkmark \end{array}$$

EXAMPLE 6.59

Simplify: ⓐ $\frac{x^9}{x^7}$ ⓑ $\frac{3^{10}}{3^2}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 9 > 7, there are more factors of *x* in the numerator.	$\frac{x^9}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	x^{9-7}
Simplify.	x^2

ⓑ

Since 10 > 2, there are more factors of *x* in the numerator.	$\frac{3^{10}}{3^2}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	3^{10-2}
Simplify.	3^8

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

TRY IT : : 6.117 Simplify: ⓐ $\frac{x^{15}}{x^{10}}$ ⓑ $\frac{6^{14}}{6^5}$.

TRY IT : : 6.118

Simplify: ⓐ $\frac{y^{43}}{y^{37}}$ ⓑ $\frac{10^{15}}{10^{7}}$.

EXAMPLE 6.60

Simplify: ⓐ $\frac{b^8}{b^{12}}$ ⓑ $\frac{7^3}{7^5}$.

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

ⓐ

Since 12 > 8, there are more factors of *b* in the denominator.	$\frac{b^8}{b^{12}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{b^{12-8}}$
Simplify.	$\frac{1}{b^4}$

ⓑ

Since 5 > 3, there are more factors of 3 in the denominator.	$\frac{7^3}{7^5}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{7^{5-3}}$
Simplify.	$\frac{1}{7^2}$
Simplify.	$\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

TRY IT : : 6.119

Simplify: ⓐ $\frac{x^{18}}{x^{22}}$ ⓑ $\frac{12^{15}}{12^{30}}$.

TRY IT : : 6.120

Simplify: ⓐ $\frac{m^7}{m^{15}}$ ⓑ $\frac{9^8}{9^{19}}$.

Notice the difference in the two previous examples:

- If we start with more factors in the numerator, we will end up with factors in the numerator.
- If we start with more factors in the denominator, we will end up with factors in the denominator.

The first step in simplifying an expression using the Quotient Property for Exponents is to determine whether the exponent is larger in the numerator or the denominator.

EXAMPLE 6.61

Simplify: ⓐ $\frac{a^5}{a^9}$ ⓑ $\frac{x^{11}}{x^7}$.

Solution

ⓐ Is the exponent of a larger in the numerator or denominator? Since 9 > 5, there are more a's in the denominator and so we will end up with factors in the denominator.

	$\frac{a^5}{a^9}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{a^{9-5}}$
Simplify.	$\frac{1}{a^4}$

ⓑ Notice there are more factors of x in the numerator, since 11 > 7. So we will end up with factors in the numerator.

	$\frac{x^{11}}{x^7}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	x^{11-7}
Simplify.	x^4

TRY IT : : 6.121 Simplify: ⓐ $\frac{b^{19}}{b^{11}}$ ⓑ $\frac{z^5}{z^{11}}$.

TRY IT : : 6.122 Simplify: ⓐ $\frac{p^9}{p^{17}}$ ⓑ $\frac{w^{13}}{w^9}$.

Simplify Expressions with an Exponent of Zero

A special case of the Quotient Property is when the exponents of the numerator and denominator are equal, such as an expression like $\frac{a^m}{a^m}$. From your earlier work with fractions, you know that:

$$\frac{2}{2} = 1 \qquad \frac{17}{17} = 1 \qquad \frac{-43}{-43} = 1$$

In words, a number divided by itself is 1. So, $\frac{x}{x} = 1$, for any $x\ (x \neq 0)$, since any number divided by itself is 1.

The Quotient Property for Exponents shows us how to simplify $\frac{a^m}{a^n}$ when $m > n$ and when $n < m$ by subtracting exponents. What if $m = n$?

Consider $\frac{8}{8}$, which we know is 1.

	$\frac{8}{8} = 1$
Write 8 as 2^3.	$\frac{2^3}{2^3} = 1$
Subtract exponents.	$2^{3-3} = 1$
Simplify.	$2^0 = 1$

Now we will simplify $\frac{a^m}{a^m}$ in two ways to lead us to the definition of the zero exponent. In general, for $a \neq 0$:

$\frac{a^m}{a^m}$	$\frac{a^m}{a^m}$
a^{m-m}	$\frac{\overbrace{\not{a}\cdot\not{a}\cdot\ldots\cdot\not{a}}^{m\text{ factors}}}{\underbrace{\not{a}\cdot\not{a}\cdot\ldots\cdot\not{a}}_{m\text{ factors}}}$
a^0	1

We see $\frac{a^m}{a^m}$ simplifies to a^0 and to 1. So $a^0 = 1$.

Zero Exponent

If a is a non-zero number, then $a^0 = 1$.

Any nonzero number raised to the zero power is 1.

In this text, we assume any variable that we raise to the zero power is not zero.

EXAMPLE 6.62

Simplify: ⓐ 9^0 ⓑ n^0.

Solution

The definition says any non-zero number raised to the zero power is 1.

ⓐ

	9^0
Use the definition of he zero exponent.	1

ⓑ

	n^0
Use the definition of he zero exponent.	1

TRY IT : : 6.123 Simplify: ⓐ 15^0 ⓑ m^0.

TRY IT : : 6.124 Simplify: ⓐ k^0 ⓑ 29^0.

Now that we have defined the zero exponent, we can expand all the Properties of Exponents to include whole number exponents.

What about raising an expression to the zero power? Let's look at $(2x)^0$. We can use the product to a power rule to rewrite this expression.

	$(2x)^0$
Use the product to a power rule.	$2^0 x^0$
Use the zero exponent property.	$1 \cdot 1$
Simplify.	1

This tells us that any nonzero expression raised to the zero power is one.

EXAMPLE 6.63

Simplify: ⓐ $(5b)^0$ ⓑ $\left(-4a^2 b\right)^0$.

Solution

ⓐ

	$(5b)^0$
Use the definition of he zero exponent.	1

ⓑ

	$\left(-4a^2b\right)^0$
Use the definition of he zero exponent.	1

TRY IT : : 6.125 Simplify: ⓐ $(11z)^0$ ⓑ $\left(-11pq^3\right)^0$.

TRY IT : : 6.126 Simplify: ⓐ $(-6d)^0$ ⓑ $\left(-8m^2n^3\right)^0$.

Simplify Expressions Using the Quotient to a Power Property

Now we will look at an example that will lead us to the Quotient to a Power Property.

	$\left(\frac{x}{y}\right)^3$
This means:	$\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$\frac{x\cdot x\cdot x}{y\cdot y\cdot y}$
Write with exponents.	$\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

We write:	$\left(\frac{x}{y}\right)^3$
	$\frac{x^3}{y^3}$

This leads to the *Quotient to a Power Property for Exponents*.

Quotient to a Power Property for Exponents

If a and b are real numbers, $b \neq 0$, and m is a counting number, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.

An example with numbers may help you understand this property:

$$\begin{aligned}\left(\frac{2}{3}\right)^3 &= \frac{2^3}{3^3}\\ \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} &= \frac{8}{27}\\ \frac{8}{27} &= \frac{8}{27}\ \checkmark\end{aligned}$$

EXAMPLE 6.64

Simplify: ⓐ $\left(\frac{3}{7}\right)^2$ ⓑ $\left(\frac{b}{3}\right)^4$ ⓒ $\left(\frac{k}{j}\right)^3$.

Solution

ⓐ

	$\left(\frac{3}{7}\right)^2$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{3^2}{7^2}$
Simplify.	$\frac{9}{49}$

ⓑ

	$\left(\frac{b}{3}\right)^4$
Use the Quotient Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{b^4}{3^4}$
Simplify.	$\frac{b^4}{81}$

ⓒ

	$\left(\frac{k}{j}\right)^3$
Raise the numerator and denominator to the third power.	$\frac{k^3}{j^3}$

TRY IT : : 6.127 Simplify: ⓐ $\left(\frac{5}{8}\right)^2$ ⓑ $\left(\frac{p}{10}\right)^4$ ⓒ $\left(\frac{m}{n}\right)^7$.

TRY IT : : 6.128 Simplify: ⓐ $\left(\frac{1}{3}\right)^3$ ⓑ $\left(\frac{-2}{q}\right)^3$ ⓒ $\left(\frac{w}{x}\right)^4$.

Simplify Expressions by Applying Several Properties

We'll now summarize all the properties of exponents so they are all together to refer to as we simplify expressions using several properties. Notice that they are now defined for whole number exponents.

Summary of Exponent Properties

If a and b are real numbers, and m and n are whole numbers, then

Product Property	$a^m \cdot a^n$	$= a^{m+n}$
Power Property	$(a^m)^n$	$= a^{m \cdot n}$
Product to a Power	$(ab)^m$	$= a^m b^m$
Quotient Property	$\frac{a^m}{b^m}$	$= a^{m-n},\ a \neq 0,\ m > n$
	$\frac{a^m}{a^n}$	$= \frac{1}{a^{n-m}},\ a \neq 0,\ n > m$
Zero Exponent Definitio	a^o	$= 1,\ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$= \frac{a^m}{b^m},\ b \neq 0$

EXAMPLE 6.65

Simplify: $\frac{\left(y^4\right)^2}{y^6}$.

Solution

	$\frac{\left(y^4\right)^2}{y^6}$
Multiply the exponents in the numerator.	$\frac{y^8}{y^6}$
Subtract the exponents.	y^2

TRY IT : : 6.129

Simplify: $\frac{\left(m^5\right)^4}{m^7}$.

TRY IT : : 6.130

Simplify: $\frac{\left(k^2\right)^6}{k^7}$.

EXAMPLE 6.66

Simplify: $\frac{b^{12}}{\left(b^2\right)^6}$.

Solution

	$\frac{b^{12}}{\left(b^2\right)^6}$
Multiply the exponents in the denominator.	$\frac{b^{12}}{b^{12}}$
Subtract the exponents.	b^0
Simplify.	1

Notice that after we simplified the denominator in the first step, the numerator and the denominator were equal. So the final value is equal to 1.

TRY IT : : 6.131 Simplify: $\dfrac{n^{12}}{(n^3)^4}$.

TRY IT : : 6.132 Simplify: $\dfrac{x^{15}}{(x^3)^5}$.

EXAMPLE 6.67

Simplify: $\left(\dfrac{y^9}{y^4}\right)^2$.

Solution

	$\left(\dfrac{y^9}{y^4}\right)^2$
Remember parentheses come before exponents. Notice the bases are the same, so we can simplify inside the parentheses. Subtract the exponents.	$(y^5)^2$
Multiply the exponents.	y^{10}

TRY IT : : 6.133 Simplify: $\left(\dfrac{r^5}{r^3}\right)^4$.

TRY IT : : 6.134 Simplify: $\left(\dfrac{v^6}{v^4}\right)^3$.

EXAMPLE 6.68

Simplify: $\left(\dfrac{j^2}{k^3}\right)^4$.

Solution

Here we cannot simplify inside the parentheses first, since the bases are not the same.

	$\left(\dfrac{j^2}{k^3}\right)^4$
Raise the numerator and denominator to the third power using the Quotient to a Power Property, $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.	$\dfrac{(j^2)^4}{(k^3)^4}$
Use the Power Property and simplify.	$\dfrac{j^8}{k^{12}}$

TRY IT : : 6.135 Simplify: $\left(\dfrac{a^3}{b^2}\right)^4$.

TRY IT : : 6.136

Simplify: $\left(\frac{q^7}{r^5}\right)^3$.

EXAMPLE 6.69

Simplify: $\left(\frac{2m^2}{5n}\right)^4$.

Solution

	$\left(\frac{2m^2}{5n}\right)^4$
Raise the numerator and denominator to the fourth power, using the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$\frac{(2m^2)^4}{(5n)^4}$
Raise each factor to the fourth power.	$\frac{2^4(m^2)^4}{5^4n^4}$
Use the Power Property and simplify.	$\frac{16m^8}{625n^4}$

TRY IT : : 6.137

Simplify: $\left(\frac{7x^3}{9y}\right)^2$.

TRY IT : : 6.138

Simplify: $\left(\frac{3x^4}{7y}\right)^2$.

EXAMPLE 6.70

Simplify: $\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$.

Solution

	$\frac{\left(x^3\right)^4\left(x^2\right)^5}{\left(x^6\right)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\frac{\left(x^{12}\right)\left(x^{10}\right)}{\left(x^{30}\right)}$
Add the exponents in the numerator.	$\frac{x^{22}}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.	$\frac{1}{x^8}$

TRY IT : : 6.139

Simplify: $\frac{\left(a^2\right)^3\left(a^2\right)^4}{\left(a^4\right)^5}$.

TRY IT : : 6.140

Simplify: $\dfrac{\left(p^3\right)^4\left(p^5\right)^3}{\left(p^7\right)^6}$.

EXAMPLE 6.71

Simplify: $\dfrac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$.

Solution

	$\dfrac{\left(10p^3\right)^2}{(5p)^3\left(2p^5\right)^4}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$\dfrac{(10)^2\left(p^3\right)^2}{(5)^3(p)^3(2)^4\left(p^5\right)^4}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$\dfrac{100p^6}{125p^3 \cdot 16p^{20}}$
Add the exponents in the denominator.	$\dfrac{100p^6}{125 \cdot 16p^{23}}$
Use the Quotient Property, $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$.	$\dfrac{100}{125 \cdot 16p^{17}}$
Simplify.	$\dfrac{1}{20p^{17}}$

TRY IT : : 6.141

Simplify: $\dfrac{\left(3r^3\right)^2\left(r^3\right)^7}{\left(r^3\right)^3}$.

TRY IT : : 6.142

Simplify: $\dfrac{\left(2x^4\right)^5}{\left(4x^3\right)^2\left(x^3\right)^5}$.

Divide Monomials

You have now been introduced to all the properties of exponents and used them to simplify expressions. Next, you'll see how to use these properties to divide monomials. Later, you'll use them to divide polynomials.

EXAMPLE 6.72

Find the quotient: $56x^7 \div 8x^3$.

Solution

	$56x^7 \div 8x^3$
Rewrite as a fraction.	$\frac{56x^7}{8x^3}$
Use fraction multiplication.	$\frac{56}{8} \cdot \frac{x^7}{x^3}$
Simplify and use the Quotient Property.	$7x^4$

TRY IT : : 6.143 Find the quotient: $42y^9 \div 6y^3$.

TRY IT : : 6.144 Find the quotient: $48z^8 \div 8z^2$.

EXAMPLE 6.73

Find the quotient: $\frac{45a^2b^3}{-5ab^5}$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$\frac{45a^2b^3}{-5ab^5}$
Use fraction multiplication.	$\frac{45}{-5} \cdot \frac{a^2}{a} \cdot \frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$-9 \cdot a \cdot \frac{1}{b^2}$
Multiply.	$-\frac{9a}{b^2}$

TRY IT : : 6.145 Find the quotient: $\frac{-72a^7b^3}{8a^{12}b^4}$.

TRY IT : : 6.146 Find the quotient: $\frac{-63c^8d^3}{7c^{12}d^2}$.

EXAMPLE 6.74

Find the quotient: $\frac{24a^5b^3}{48ab^4}$.

Solution

	$\frac{24a^5b^3}{48ab^4}$
Use fraction multiplication.	$\frac{24}{48} \cdot \frac{a^5}{a} \cdot \frac{b^3}{b^4}$
Simplify and use the Quotient Property.	$\frac{1}{2} \cdot a^4 \cdot \frac{1}{b}$
Multiply.	$\frac{a^4}{2b}$

TRY IT : : 6.147 Find the quotient: $\frac{16a^7b^6}{24ab^8}$.

TRY IT : : 6.148 Find the quotient: $\frac{27p^4q^7}{-45p^{12}q}$.

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

EXAMPLE 6.75

Find the quotient: $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$\frac{2y^6}{3x^4}$

TRY IT : : 6.149 Find the quotient: $\frac{28x^5y^{14}}{49x^9y^{12}}$.

TRY IT : : 6.150 Find the quotient: $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

In all examples so far, there was no work to do in the numerator or denominator before simplifying the fraction. In the next example, we'll first find the product of two monomials in the numerator before we simplify the fraction. This follows the order of operations. Remember, a fraction bar is a grouping symbol.

EXAMPLE 6.76

Find the quotient: $\frac{(6x^2y^3)(5x^3y^2)}{(3x^4y^5)}$.

Solution

	$\frac{(6x^2y^3)(5x^3y^2)}{(3x^4y^5)}$
Simplify the numerator.	$\frac{30x^5y^5}{3x^4y^5}$
Simplify.	$10x$

TRY IT : : 6.151 Find the quotient: $\frac{(6a^4b^5)(4a^2b^5)}{12a^5b^8}$.

TRY IT : : 6.152

Find the quotient: $\frac{\left(-12x^6y^9\right)\left(-4x^5y^8\right)}{-12x^{10}y^{12}}$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing monomials:

- **Rational Expressions (https://openstax.org/l/25RationalExp)**
- **Dividing Monomials (https://openstax.org/l/25DivideMono)**
- **Dividing Monomials 2 (https://openstax.org/l/25DivideMono2)**

6.5 EXERCISES

Practice Makes Perfect

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

356. ⓐ $\frac{x^{18}}{x^3}$ ⓑ $\frac{5^{12}}{5^3}$

357. ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^2}$

358. ⓐ $\frac{p^{21}}{p^7}$ ⓑ $\frac{4^{16}}{4^4}$

359. ⓐ $\frac{u^{24}}{u^3}$ ⓑ $\frac{9^{15}}{9^5}$

360. ⓐ $\frac{q^{18}}{q^{36}}$ ⓑ $\frac{10^2}{10^3}$

361. ⓐ $\frac{t^{10}}{t^{40}}$ ⓑ $\frac{8^3}{8^5}$

362. ⓐ $\frac{b}{b^9}$ ⓑ $\frac{4}{4^6}$

363. ⓐ $\frac{x}{x^7}$ ⓑ $\frac{10}{10^3}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

364.
ⓐ 20^0
ⓑ b^0

365.
ⓐ 13^0
ⓑ k^0

366.
ⓐ -27^0
ⓑ $-\left(27^0\right)$

367.
ⓐ -15^0
ⓑ $-\left(15^0\right)$

368.
ⓐ $(25x)^0$
ⓑ $25x^0$

369.
ⓐ $(6y)^0$
ⓑ $6y^0$

370.
ⓐ $(12x)^0$
ⓑ $\left(-56p^4q^3\right)^0$

371.
ⓐ $7y^0\ (17y)^0$
ⓑ $\left(-93c^7d^{15}\right)^0$

372.
ⓐ $12n^0 - 18m^0$
ⓑ $(12n)^0 - (18m)^0$

373.
ⓐ $15r^0 - 22s^0$
ⓑ $(15r)^0 - (22s)^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

374. ⓐ $\left(\frac{3}{4}\right)^3$ ⓑ $\left(\frac{p}{2}\right)^5$ ⓒ $\left(\frac{x}{y}\right)^6$

375. ⓐ $\left(\frac{2}{5}\right)^2$ ⓑ $\left(\frac{x}{3}\right)^4$ ⓒ $\left(\frac{a}{b}\right)^5$

376. ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^2$

377. ⓐ $\left(\frac{x}{2y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^4$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

378. $\dfrac{(a^2)^3}{a^4}$

379. $\dfrac{(p^3)^4}{p^5}$

380. $\dfrac{(y^3)^4}{y^{10}}$

381. $\dfrac{(x^4)^5}{x^{15}}$

382. $\dfrac{u^6}{(u^3)^2}$

383. $\dfrac{v^{20}}{(v^4)^5}$

384. $\dfrac{m^{12}}{(m^8)^3}$

385. $\dfrac{n^8}{(n^6)^4}$

386. $\left(\dfrac{p^9}{p^3}\right)^5$

387. $\left(\dfrac{q^8}{q^2}\right)^3$

388. $\left(\dfrac{r^2}{r^6}\right)^3$

389. $\left(\dfrac{m^4}{m^7}\right)^4$

390. $\left(\dfrac{p}{r^{11}}\right)^2$

391. $\left(\dfrac{a}{b^6}\right)^3$

392. $\left(\dfrac{w^5}{x^3}\right)^8$

393. $\left(\dfrac{y^4}{z^{10}}\right)^5$

394. $\left(\dfrac{2j^3}{3k}\right)^4$

395. $\left(\dfrac{3m^5}{5n}\right)^3$

396. $\left(\dfrac{3c^2}{4d^6}\right)^3$

397. $\left(\dfrac{5u^7}{2v^3}\right)^4$

398. $\left(\dfrac{k^2k^8}{k^3}\right)^2$

399. $\left(\dfrac{j^2j^5}{j^4}\right)^3$

400. $\dfrac{(t^2)^5(t^4)^2}{(t^3)^7}$

401. $\dfrac{(q^3)^6(q^2)^3}{(q^4)^8}$

402. $\dfrac{(-2p^2)^4(3p^4)^2}{(-6p^3)^2}$

403. $\dfrac{(-2k^3)^2(6k^2)^4}{(9k^4)^2}$

404. $\dfrac{(-4m^3)^2(5m^4)^3}{(-10m^6)^3}$

405. $\dfrac{(-10n^2)^3(4n^5)^2}{(2n^8)^2}$

Divide Monomials

In the following exercises, divide the monomials.

406. $56b^8 \div 7b^2$

407. $63v^{10} \div 9v^2$

408. $-88y^{15} \div 8y^3$

409. $-72u^{12} \div 12u^4$

410. $\dfrac{45a^6b^8}{-15a^{10}b^2}$

411. $\dfrac{54x^9y^3}{-18x^6y^{15}}$

412. $\frac{15r^4 s^9}{18r^9 s^2}$

413. $\frac{20m^8 n^4}{30m^5 n^9}$

414. $\frac{18a^4 b^8}{-27a^9 b^5}$

415. $\frac{45x^5 y^9}{-60x^8 y^6}$

416. $\frac{64q^{11} r^9 s^3}{48q^6 r^8 s^5}$

417. $\frac{65a^{10} b^8 c^5}{42a^7 b^6 c^8}$

418. $\frac{(10m^5 n^4)(5m^3 n^6)}{25m^7 n^5}$

419. $\frac{(-18p^4 q^7)(-6p^3 q^8)}{-36p^{12} q^{10}}$

420. $\frac{(6a^4 b^3)(4ab^5)}{(12a^2 b)(a^3 b)}$

421. $\frac{(4u^2 v^5)(15u^3 v)}{(12u^3 v)(u^4 v)}$

Mixed Practice

422.

ⓐ $24a^5 + 2a^5$

ⓑ $24a^5 - 2a^5$

ⓒ $24a^5 \cdot 2a^5$

ⓓ $24a^5 \div 2a^5$

423.

ⓐ $15n^{10} + 3n^{10}$

ⓑ $15n^{10} - 3n^{10}$

ⓒ $15n^{10} \cdot 3n^{10}$

ⓓ $15n^{10} \div 3n^{10}$

424.

ⓐ $p^4 \cdot p^6$

ⓑ $(p^4)^6$

425.

ⓐ $q^5 \cdot q^3$

ⓑ $(q^5)^3$

426.

ⓐ $\frac{y^3}{y}$

ⓑ $\frac{y}{y^3}$

427.

ⓐ $\frac{z^6}{z^5}$

ⓑ $\frac{z^5}{z^6}$

428. $(8x^5)(9x) \div 6x^3$

429. $(4y)(12y^7) \div 8y^2$

430. $\frac{27a^7}{3a^3} + \frac{54a^9}{9a^5}$

431. $\frac{32c^{11}}{4c^5} + \frac{42c^9}{6c^3}$

432. $\frac{32y^5}{8y^2} - \frac{60y^{10}}{5y^7}$

433. $\frac{48x^6}{6x^4} - \frac{35x^9}{7x^7}$

434. $\frac{63r^6 s^3}{9r^4 s^2} - \frac{72r^2 s^2}{6s}$

435. $\frac{56y^4 z^5}{7y^3 z^3} - \frac{45y^2 z^2}{5y}$

Everyday Math

436. Memory One megabyte is approximately 10^6 bytes. One gigabyte is approximately 10^9 bytes. How many megabytes are in one gigabyte?

437. Memory One gigabyte is approximately 10^9 bytes. One terabyte is approximately 10^{12} bytes. How many gigabytes are in one terabyte?

Writing Exercises

438. Jennifer thinks the quotient $\frac{a^{24}}{a^6}$ simplifies to a^4. What is wrong with her reasoning?

439. Maurice simplifies the quotient $\frac{d^7}{d}$ by writing $\frac{\not{d}^7}{\not{d}} = 7$. What is wrong with his reasoning?

440. When Drake simplified -3^0 and $(-3)^0$ he got the same answer. Explain how using the Order of Operations correctly gives different answers.

441. Robert thinks x^0 simplifies to 0. What would you say to convince Robert he is wrong?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions using the Quotient Property for Exponents.			
simplify expressions with zero exponents.			
simplify expressions using the Quotient to a Power Property.			
simplify expressions by applying several properties.			
divide monomials.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

6.6 Divide Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Divide a polynomial by a monomial
- Divide a polynomial by a binomial

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{3}{d} + \frac{x}{d}$.
   If you missed this problem, review **Example 1.77**.
2. Simplify: $\frac{30xy^3}{5xy}$.
   If you missed this problem, review **Example 6.72**.
3. Combine like terms: $8a^2 + 12a + 1 + 3a^2 - 5a + 4$.
   If you missed this problem, review **Example 1.24**.

Divide a Polynomial by a Monomial

In the last section, you learned how to divide a monomial by a monomial. As you continue to build up your knowledge of polynomials the next procedure is to divide a polynomial of two or more terms by a monomial.

The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum,	$\frac{y}{5} + \frac{2}{5}$,
simplifies o	$\frac{y+2}{5}$.

Now we will do this in reverse to split a single fraction into separate fractions.

We'll state the fraction addition property here just as you learned it and in reverse.

Fraction Addition

If $a, b,$ and c are numbers where $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

We use the form on the left to add fractions and we use the form on the right to divide a polynomial by a monomial.

For example,	$\frac{y+2}{5}$
can be written	$\frac{y}{5} + \frac{2}{5}$.

We use this form of fraction addition to divide polynomials by monomials.

Division of a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

EXAMPLE 6.77

Find the quotient: $\frac{7y^2 + 21}{7}$.

Solution

	$\frac{7y^2 + 21}{7}$
Divide each term of the numerator by the denominator.	$\frac{7y^2}{7} + \frac{21}{7}$
Simplify each fraction.	$y^2 + 3$

TRY IT : : 6.153 Find the quotient: $\frac{8z^2 + 24}{4}$.

TRY IT : : 6.154 Find the quotient: $\frac{18z^2 - 27}{9}$.

Remember that division can be represented as a fraction. When you are asked to divide a polynomial by a monomial and it is not already in fraction form, write a fraction with the polynomial in the numerator and the monomial in the denominator.

EXAMPLE 6.78

Find the quotient: $\left(18x^3 - 36x^2\right) \div 6x$.

Solution

	$\left(18x^3 - 36x^2\right) \div 6x$
Rewrite as a fraction.	$\frac{18x^3 - 36x^2}{6x}$
Divide each term of the numerator by the denominator.	$\frac{18x^3}{6x} - \frac{36x^2}{6x}$
Simplify.	$3x^2 - 6x$

TRY IT : : 6.155 Find the quotient: $\left(27b^3 - 33b^2\right) \div 3b$.

TRY IT : : 6.156 Find the quotient: $\left(25y^3 - 55y^2\right) \div 5y$.

When we divide by a negative, we must be extra careful with the signs.

EXAMPLE 6.79

Find the quotient: $\frac{12d^2 - 16d}{-4}$.

Solution

	$\frac{12d^2 - 16d}{-4}$
Divide each term of the numerator by the denominator.	$\frac{12d^2}{-4} - \frac{16d}{-4}$
Simplify. Remember, subtracting a negative is like adding a positive!	$-3d^2 + 4d$

TRY IT : : 6.157 Find the quotient: $\frac{25y^2 - 15y}{-5}$.

TRY IT : : 6.158 Find the quotient: $\frac{42b^2 - 18b}{-6}$.

EXAMPLE 6.80

Find the quotient: $\frac{105y^5 + 75y^3}{5y^2}$.

Solution

	$\frac{105y^5 + 75y^3}{5y^2}$
Separate the terms.	$\frac{105y^5}{5y^2} + \frac{75y^3}{5y^2}$
Simplify.	$21y^3 + 15y$

TRY IT : : 6.159 Find the quotient: $\frac{60d^7 + 24d^5}{4d^3}$.

TRY IT : : 6.160 Find the quotient: $\frac{216p^7 - 48p^5}{6p^3}$.

EXAMPLE 6.81

Find the quotient: $\left(15x^3y - 35xy^2\right) \div (-5xy)$.

Solution

	$\left(15x^3y - 35xy^2\right) \div (-5xy)$
Rewrite as a fraction.	$\frac{15x^3y - 35xy^2}{-5xy}$
Separate the terms. Be careful with the signs!	$\frac{15x^3y}{-5xy} - \frac{35xy^2}{-5xy}$
Simplify.	$-3x^2 + 7y$

TRY IT : : 6.161 Find the quotient: $\left(32a^2b - 16ab^2\right) \div (-8ab)$.

TRY IT : : 6.162 Find the quotient: $\left(-48a^8b^4 - 36a^6b^5\right) \div \left(-6a^3b^3\right)$.

EXAMPLE 6.82

Find the quotient: $\frac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$.

Solution

	$\frac{36x^3y^2 + 27x^2y^2 - 9x^2y^3}{9x^2y}$
Separate the terms.	$\frac{36x^3y^2}{9x^2y} + \frac{27x^2y^2}{9x^2y} - \frac{9x^2y^3}{9x^2y}$
Simplify.	$4xy + 3y - y^2$

TRY IT : : 6.163 Find the quotient: $\frac{40x^3y^2 + 24x^2y^2 - 16x^2y^3}{8x^2y}$.

TRY IT : : 6.164 Find the quotient: $\frac{35a^4b^2 + 14a^4b^3 - 42a^2b^4}{7a^2b^2}$.

EXAMPLE 6.83

Find the quotient: $\frac{10x^2 + 5x - 20}{5x}$.

Solution

	$\frac{10x^2 + 5x - 20}{5x}$
Separate the terms.	$\frac{10x^2}{5x} + \frac{5x}{5x} - \frac{20}{5x}$
Simplify.	$2x + 1 + \frac{4}{x}$

TRY IT : : 6.165 Find the quotient: $\frac{18c^2 + 6c - 9}{6c}$.

TRY IT : : 6.166 Find the quotient: $\frac{10d^2 - 5d - 2}{5d}$.

Divide a Polynomial by a Binomial

To divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let’s look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

We write the long division	$25\overline{)875}$
We divide the first two digits, 87, by 25.	$\begin{array}{r} 3 \\ 25\overline{)875} \end{array}$
We multiply 3 times 25 and write the product under the 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{75} \end{array}$
Now we subtract 75 from 87.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{-75} \\ 12 \end{array}$
Then we bring down the third digit of the dividend, 5.	$\begin{array}{r} 3 \\ 25\overline{)875} \\ \underline{-75} \\ 125 \end{array}$
Repeat the process, dividing 25 into 125.	$\begin{array}{r} 35 \\ 25\overline{)875} \\ \underline{-75} \\ 125 \\ \underline{-125} \end{array}$

We check division by multiplying the quotient by the divisor.

If we did the division correctly, the product should equal the dividend.

$$\begin{array}{c} 35 \cdot 25 \\ 875\checkmark \end{array}$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

EXAMPLE 6.84

Find the quotient: $\left(x^2 + 9x + 20\right) \div (x + 5)$.

Solution

	$(x^2 + 9x + 20) \div (x + 5)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x + 5\overline{)\ x^2 + 9x + 20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?"	
Put the answer, x, in the quotient over the x term.	$\begin{array}{l} x \\ x + 5\overline{)\ x^2 + 9x + 20} \end{array}$

Multiply x times $x + 5$. Line up the like terms under the dividend.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{x^2+5x} \end{array}$
Subtract $x^2 + 5x$ from $x^2 + 9x$.	
You may find it easier to change the signs and then add. Then bring down the last term, 20.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, "What do I need to multiply x by to get $4x$?"	
Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \end{array}$
Multiply 4 times $x + 5$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{4x+20} \end{array}$
Subtract $4x + 20$ from $4x + 20$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{-4x+(-20)} \\ 0 \end{array}$
Check:	
Multiply the quotient by the divisor.	
$(x + 4)(x + 5)$	
You should get the dividend.	
$x^2 + 9x + 20$✓	

TRY IT : : 6.167 Find the quotient: $\left(y^2 + 10y + 21\right) \div (y + 3)$.

TRY IT : : 6.168 Find the quotient: $\left(m^2 + 9m + 20\right) \div (m + 4)$.

When the divisor has subtraction sign, we must be extra careful when we multiply the partial quotient and then subtract. It may be safer to show that we change the signs and then add.

EXAMPLE 6.85

Find the quotient: $\left(2x^2 - 5x - 3\right) \div (x - 3)$.

Solution

	$(2x^2-5x-3)\div(x-3)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x-3\overline{)2x^2-5x-3}$
Divide $2x^2$ by x. Put the answer, $2x$, in the quotient over the x term.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \end{array}$
Multiply $2x$ times $x-3$. Line up the like terms under the dividend.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{2x^2-6x} \end{array}$
Subtract $2x^2-6x$ from $2x^2-5x$. Change the signs and then add. Then bring down the last term.	$\begin{array}{r} 2x \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \end{array}$
Divide x by x. Put the answer, 1, in the quotient over the constant term.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2-(-6x)} \\ x-3 \end{array}$
Multiply 1 times $x-3$.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \\ \underline{x-3} \end{array}$
Subtract $x-3$ from $x-3$ by changing the signs and adding.	$\begin{array}{r} 2x+1 \\ x-3\overline{)2x^2-5x-3} \\ \underline{-2x^2+6x} \\ x-3 \\ \underline{-x+3} \\ 0 \end{array}$
To check, multiply $(x-3)(2x+1)$.	
The result should be $2x^2-5x-3$.	

TRY IT : : 6.169 Find the quotient: $(2x^2-3x-20)\div(x-4)$.

TRY IT : : 6.170 Find the quotient: $(3x^2-16x-12)\div(x-6)$.

When we divided 875 by 25, we had no remainder. But sometimes division of numbers does leave a remainder. The same is true when we divide polynomials. In **Example 6.86**, we'll have a division that leaves a remainder. We write the remainder as a fraction with the divisor as the denominator.

EXAMPLE 6.86

Find the quotient: $(x^3-x^2+x+4)\div(x+1)$.

Solution

	$(x^3 - x^2 + x + 4) \div (x + 1)$
Write it as a long division problem.	
Be sure the dividend is in standard form.	$x + 1 \overline{)\, x^3 - x^2 + x + 4}$
Divide x^3 by x. Put the answer, x^2, in the quotient over the x^2 term. Multiply x^2 times $x + 1$. Line up the like terms under the dividend.	x^2 $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{x^3 + x^2}$
Subtract $x^3 + x^2$ from $x^3 - x^2$ by changing the signs and adding. Then bring down the next term.	x^2 $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{-x^3 + (-x^2)}$ $-2x^2 + x$
Divide $-2x^2$ by x. Put the answer, $-2x$, in the quotient over the x term. Multiply $-2x$ times $x + 1$. Line up the like terms under the dividend.	$x^2 - 2x$ $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{-x^3 + (-x^2)}$ $-2x^2 + x$ $\underline{-2x^2 - 2x}$
Subtract $-2x^2 - 2x$ from $-2x^2 + x$ by changing the signs and adding. Then bring down the last term.	$x^2 - 2x$ $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{-x^3 + (-x^2)}$ $-2x^2 + x$ $\underline{+2x^2 + 2x}$ $3x + 4$
Divide $3x$ by x. Put the answer, 3, in the quotient over the constant term. Multiply 3 times $x + 1$. Line up the like terms under the dividend.	$x^2 - 2x + 3$ $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{-x^3 + (-x^2)}$ $-2x^2 + x$ $\underline{+2x^2 + 2x}$ $3x + 4$ $\underline{3x + 3}$
Subtract $3x + 3$ from $3x + 4$ by changing the signs and adding. Write the remainder as a fraction with the divisor as the denominator.	$x^2 - 2x + 3 + \frac{1}{x + 1}$ $x + 1 \overline{)\, x^3 - x^2 + x + 4}$ $\underline{-x^3 + (-x^2)}$ $-2x^2 + x$ $\underline{+2x^2 + 2x}$ $3x + 4$ $\underline{-3x + (-3)}$ 1
To check, multiply $(x + 1)\left(x^2 - 2x + 3 + \frac{1}{x + 1}\right)$. The result should be $x^3 - x^2 + x + 4$.	

TRY IT : : 6.171 Find the quotient: $\left(x^3 + 5x^2 + 8x + 6\right) \div (x + 2)$.

TRY IT : : 6.172 Find the quotient: $\left(2x^3 + 8x^2 + x - 8\right) \div (x + 1)$.

Look back at the dividends in **Example 6.84**, **Example 6.85**, and **Example 6.86**. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in **Example 6.87** will be $x^4 - x^2 + 5x - 2$. It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

EXAMPLE 6.87

Find the quotient: $\left(x^4 - x^2 + 5x - 2\right) \div (x + 2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

	$(x^4 - x^2 + 5x - 2) \div (x + 2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x + 2\,\overline{)\,x^4 - 0x^3 - x^2 + 5x - 2}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 times $x + 2$. Line up the like terms. Subtract and then bring down the next term.	Quotient: x^3 $x + 2\,\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ It may be helpful to change the signs and add.
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	Quotient: $x^3 - 2x^2$ $x + 2\,\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ It may be helpful to change the signs and add.
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x + 1$. Line up the like terms. Subtract and bring down the next term.	Quotient: $x^3 - 2x^2 + 3x$ $x + 2\,\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 2$ It may be helpful to change the signs and add.
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x + 1$. Line up the like terms. Change the signs, add.	Quotient: $x^3 - 2x^2 + 3x - 1$ $x + 2\,\overline{)\,x^4 + 0x^3 - x^2 + 5x - 2}$ $-(x^4 + 2x^3)$ $-2x^3 - x^2$ $-(-2x^3 - 4x^2)$ $3x^2 + 5x$ $-(3x^2 + 6x)$ $-x - 2$ $-(-x - 2)$ 0 It may be helpful to change the signs and add.
To check, multiply $(x + 2)\left(x^3 - 2x^2 + 3x - 1\right)$.	

The result should be $x^4 - x^2 + 5x - 2$.

> **TRY IT : :** 6.173 Find the quotient: $\left(x^3 + 3x + 14\right) \div (x + 2)$.

> **TRY IT : :** 6.174 Find the quotient: $\left(x^4 - 3x^3 - 1000\right) \div (x + 5)$.

In Example 6.88, we will divide by $2a - 3$. As we divide we will have to consider the constants as well as the variables.

EXAMPLE 6.88

Find the quotient: $\left(8a^3 + 27\right) \div (2a + 3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$$(8a^3 + 27) \div (2a + 3)$$

$$\begin{array}{rl} & 4a^2 - 6a + 9 \\ 2a + 3\,\big) & 8a^3 + 0a^2 + 0a + 27 \\ & \underline{-(8a^3 + 12a^2)} \quad \leftarrow 4a^2(2a+3) \\ & -12a^2 + 0a \\ & \underline{-(-12a^2 - 18a)} \quad \leftarrow 6a(2a+3) \\ & 18a + 27 \\ & \underline{-(18a + 27)} \quad \leftarrow 9(2a+3) \\ & 0 \end{array}$$

To check, multiply $(2a + 3)\left(4a^2 - 6a + 9\right)$.

The result should be $8a^3 + 27$.

> **TRY IT : :** 6.175 Find the quotient: $\left(x^3 - 64\right) \div (x - 4)$.

> **TRY IT : :** 6.176 Find the quotient: $\left(125x^3 - 8\right) \div (5x - 2)$.

MEDIA : :

Access these online resources for additional instruction and practice with dividing polynomials:

- Divide a Polynomial by a Monomial (https://openstax.org/l/25DividePolyMo1)
- Divide a Polynomial by a Monomial 2 (https://openstax.org/l/25DividePolyMo2)
- Divide Polynomial by Binomial (https://openstax.org/l/25DividePolyBin)

6.6 EXERCISES

Practice Makes Perfect

In the following exercises, divide each polynomial by the monomial.

442. $\frac{45y+36}{9}$

443. $\frac{30b+75}{5}$

444. $\frac{8d^2-4d}{2}$

445. $\frac{42x^2-14x}{7}$

446. $\left(16y^2-20y\right) \div 4y$

447. $\left(55w^2-10w\right) \div 5w$

448. $\left(9n^4+6n^3\right) \div 3n$

449. $\left(8x^3+6x^2\right) \div 2x$

450. $\frac{18y^2-12y}{-6}$

451. $\frac{20b^2-12b}{-4}$

452. $\frac{35a^4+65a^2}{-5}$

453. $\frac{51m^4+72m^3}{-3}$

454. $\frac{310y^4-200y^3}{5y^2}$

455. $\frac{412z^8-48z^5}{4z^3}$

456. $\frac{46x^3+38x^2}{2x^2}$

457. $\frac{51y^4+42y^2}{3y^2}$

458. $\left(24p^2-33p\right) \div (-3p)$

459. $\left(35x^4-21x\right) \div (-7x)$

460. $\left(63m^4-42m^3\right) \div \left(-7m^2\right)$

461. $\left(48y^4-24y^3\right) \div \left(-8y^2\right)$

462. $\left(63a^2b^3+72ab^4\right) \div (9ab)$

463. $\left(45x^3y^4+60xy^2\right) \div (5xy)$

464. $\frac{52p^5q^4+36p^4q^3-64p^3q^2}{4p^2q}$

465. $\frac{49c^2d^2-70c^3d^3-35c^2d^4}{7cd^2}$

466. $\frac{66x^3y^2-110x^2y^3-44x^4y^3}{11x^2y^2}$

467. $\frac{72r^5s^2+132r^4s^3-96r^3s^5}{12r^2s^2}$

468. $\frac{4w^2+2w-5}{2w}$

469. $\frac{12q^2+3q-1}{3q}$

470. $\frac{10x^2+5x-4}{-5x}$

471. $\frac{20y^2+12y-1}{-4y}$

472. $\frac{36p^3+18p^2-12p}{6p^2}$

473. $\frac{63a^3-108a^2+99a}{9a^2}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

474. $\left(y^2+7y+12\right) \div (y+3)$

475. $\left(d^2+8d+12\right) \div (d+2)$

476. $\left(x^2-3x-10\right) \div (x+2)$

477. $\left(a^2-2a-35\right) \div (a+5)$

478. $\left(t^2-12t+36\right) \div (t-6)$

479. $\left(x^2-14x+49\right) \div (x-7)$

480. $\left(6m^2-19m-20\right) \div (m-4)$

481. $\left(4x^2-17x-15\right) \div (x-5)$

482. $\left(q^2+2q+20\right) \div (q+6)$

483. $\left(p^2 + 11p + 16\right) \div (p + 8)$

484. $\left(y^2 - 3y - 15\right) \div (y - 8)$

485. $\left(x^2 + 2x - 30\right) \div (x - 5)$

486. $\left(3b^3 + b^2 + 2\right) \div (b + 1)$

487. $\left(2n^3 - 10n + 24\right) \div (n + 3)$

488. $\left(2y^3 - 6y - 36\right) \div (y - 3)$

489. $\left(7q^3 - 5q - 2\right) \div (q - 1)$

490. $\left(z^3 + 1\right) \div (z + 1)$

491. $\left(m^3 + 1000\right) \div (m + 10)$

492. $\left(a^3 - 125\right) \div (a - 5)$

493. $\left(x^3 - 216\right) \div (x - 6)$

494. $\left(64x^3 - 27\right) \div (4x - 3)$

495. $\left(125y^3 - 64\right) \div (5y - 4)$

Everyday Math

496. Average cost Pictures Plus produces digital albums. The company's average cost (in dollars) to make x albums is given by the expression $\frac{7x + 500}{x}$.

ⓐ Find the quotient by dividing the numerator by the denominator.

ⓑ What will the average cost (in dollars) be to produce 20 albums?

497. Handshakes At a company meeting, every employee shakes hands with every other employee. The number of handshakes is given by the expression $\frac{n^2 - n}{2}$, where n represents the number of employees. How many handshakes will there be if there are 10 employees at the meeting?

Writing Exercises

498. James divides $48y + 6$ by 6 this way: $\frac{48y + \not{6}}{\not{6}} = 48y$. What is wrong with his reasoning?

499. Divide $\frac{10x^2 + x - 12}{2x}$ and explain with words how you get each term of the quotient.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
divide a polynomial by a monomial.			
divide a polynomial by a binomial.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

6.7 Integer Exponents and Scientific Notation

Learning Objectives

By the end of this section, you will be able to:

- Use the definition of a negative exponent
- Simplify expressions with integer exponents
- Convert from decimal notation to scientific notation
- Convert scientific notation to decimal form
- Multiply and divide using scientific notation

Be Prepared!

Before you get started, take this readiness quiz.

1. What is the place value of the 6 in the number $64,891$?
   If you missed this problem, review **Example 1.1**.
2. Name the decimal: 0.0012.
   If you missed this problem, review **Example 1.91**.
3. Subtract: $5-(-3)$.
   If you missed this problem, review **Example 1.42**.

Use the Definition of a Negative Exponent

We saw that the Quotient Property for Exponents introduced earlier in this chapter, has two forms depending on whether the exponent is larger in the numerator or the denominator.

Quotient Property for Exponents

If a is a real number, $a \neq 0$, and m and n are whole numbers, then

$$\frac{a^m}{a^n}=a^{m-n},\ m>n \quad \text{and} \quad \frac{a^m}{a^n}=\frac{1}{a^{n-m}},\ n>m$$

What if we just subtract exponents regardless of which is larger?

Let's consider $\frac{x^2}{x^5}$.

We subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^2}{x^5}$$

$$x^{2-5}$$

$$x^{-3}$$

We can also simplify $\frac{x^2}{x^5}$ by dividing out common factors:

$$\frac{\cancel{x}\cdot\cancel{x}}{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}$$

$$\frac{1}{x^3}$$

This implies that $x^{-3}=\frac{1}{x^3}$ and it leads us to the definition of a *negative exponent*.

Negative Exponent

If n is an integer and $a \neq 0$, then $a^{-n}=\frac{1}{a^n}$.

The negative exponent tells us we can re-write the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in simplest form. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

EXAMPLE 6.89

Simplify: ⓐ 4^{-2} ⓑ 10^{-3}.

Solution

ⓐ

	4^{-2}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{4^2}$
Simplify.	$\frac{1}{16}$

ⓑ

	10^{-3}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{10^3}$
Simplify.	$\frac{1}{1000}$

TRY IT : : 6.177 Simplify: ⓐ 2^{-3} ⓑ 10^{-7}.

TRY IT : : 6.178 Simplify: ⓐ 3^{-2} ⓑ 10^{-4}.

In **Example 6.89** we raised an integer to a negative exponent. What happens when we raise a fraction to a negative exponent? We'll start by looking at what happens to a fraction whose numerator is one and whose denominator is an integer raised to a negative exponent.

	$\frac{1}{a^{-n}}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\frac{1}{a^n}}$
Simplify the complex fraction.	$1 \cdot \frac{a^n}{1}$
Multiply.	a^n

This leads to the Property of Negative Exponents.

Property of Negative Exponents

If n is an integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$.

EXAMPLE 6.90

Simplify: ⓐ $\frac{1}{y^{-4}}$ ⓑ $\frac{1}{3^{-2}}$.

Solution

ⓐ

	$\frac{1}{y^{-4}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	y^4

ⓑ

	$\frac{1}{3^{-2}}$
Use the property of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	3^2
Simplify.	9

TRY IT : : 6.179 Simplify: ⓐ $\frac{1}{p^{-8}}$ ⓑ $\frac{1}{4^{-3}}$.

TRY IT : : 6.180 Simplify: ⓐ $\frac{1}{q^{-7}}$ ⓑ $\frac{1}{2^{-4}}$.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$\frac{16}{9}$
But we know that $\frac{16}{9}$ is $\left(\frac{4}{3}\right)^2$.	
This tells us that:	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Power Property*.

Quotient to a Negative Exponent Property

If a and b are real numbers, $a \neq 0,\ b \neq 0,$ and n is an integer, then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$.

EXAMPLE 6.91

Simplify: ⓐ $\left(\frac{5}{7}\right)^{-2}$ ⓑ $\left(-\frac{2x}{y}\right)^{-3}$.

Solution

ⓐ

$\left(\frac{5}{7}\right)^{-2}$

Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

Take the reciprocal of the fraction and change the sign of the exponent. $\left(\frac{7}{5}\right)^{2}$

Simplify. $\frac{49}{25}$

ⓑ

$\left(-\frac{2x}{y}\right)^{-3}$

Use the Quotient to a Negative Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.

Take the reciprocal of the fraction and change the sign of the exponent. $\left(-\frac{y}{2x}\right)^{3}$

Simplify. $-\frac{y^3}{8x^3}$

TRY IT : : 6.181 Simplify: ⓐ $\left(\frac{2}{3}\right)^{-4}$ ⓑ $\left(-\frac{6m}{n}\right)^{-2}$.

TRY IT : : 6.182 Simplify: ⓐ $\left(\frac{3}{5}\right)^{-3}$ ⓑ $\left(-\frac{a}{2b}\right)^{-4}$.

When simplifying an expression with exponents, we must be careful to correctly identify the base.

EXAMPLE 6.92

Simplify: ⓐ $(-3)^{-2}$ ⓑ -3^{-2} ⓒ $\left(-\frac{1}{3}\right)^{-2}$ ⓓ $-\left(\frac{1}{3}\right)^{-2}$.

Solution

ⓐ Here the exponent applies to the base -3 .

$(-3)^{-2}$

Take the reciprocal of the base and change the sign of the exponent. $\frac{1}{(-3)^{-2}}$

Simplify. $\frac{1}{9}$

ⓑ The expression -3^{-2} means “find the opposite of 3^{-2} ”. Here the exponent applies to the base 3.

-3^{-2}

Rewrite as a product with -1. $-1 \cdot 3^{-2}$

Take the reciprocal of the base and change the sign of the exponent. $-1 \cdot \frac{1}{3^2}$

Simplify. $-\frac{1}{9}$

ⓒ Here the exponent applies to the base $\left(-\frac{1}{3}\right)$.

	$\left(-\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$\left(-\frac{3}{1}\right)^{2}$
Simplify.	9

ⓓ The expression $-\left(\frac{1}{3}\right)^{-2}$ means "find the opposite of $\left(\frac{1}{3}\right)^{-2}$". Here the exponent applies to the base $\left(\frac{1}{3}\right)$.

	$-\left(\frac{1}{3}\right)^{-2}$
Rewrite as a product with -1.	$-1\cdot\left(\frac{1}{3}\right)^{-2}$
Take the reciprocal of the base and change the sign of the exponent.	$-1\cdot\left(\frac{3}{1}\right)^{2}$
Simplify.	-9

TRY IT : : 6.183 Simplify: ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$.

TRY IT : : 6.184 Simplify: ⓐ $(-7)^{-2}$ ⓑ -7^{-2}, ⓒ $\left(-\frac{1}{7}\right)^{-2}$ ⓓ $-\left(\frac{1}{7}\right)^{-2}$.

We must be careful to follow the Order of Operations. In the next example, parts (a) and (b) look similar, but the results are different.

EXAMPLE 6.93

Simplify: ⓐ $4\cdot 2^{-1}$ ⓑ $(4\cdot 2)^{-1}$.

Solution

ⓐ

Do exponents before multiplication.	$4\cdot 2^{-1}$
Use $a^{-n}=\frac{1}{a^n}$.	$4\cdot\frac{1}{2^1}$
Simplify.	2

ⓑ

	$(4\cdot 2)^{-1}$
Simplify inside the parentheses fir t.	$(8)^{-1}$
Use $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{8^1}$
Simplify.	$\frac{1}{8}$

TRY IT : : 6.185 Simplify: ⓐ $6\cdot 3^{-1}$ ⓑ $(6\cdot 3)^{-1}$.

TRY IT : : 6.186 Simplify: ⓐ $8\cdot 2^{-2}$ ⓑ $(8\cdot 2)^{-2}$.

When a variable is raised to a negative exponent, we apply the definition the same way we did with numbers. We will assume all variables are non-zero.

EXAMPLE 6.94

Simplify: ⓐ x^{-6} ⓑ $\left(u^4\right)^{-3}$.

Solution

ⓐ

	x^{-6}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{x^6}$

ⓑ

	$\left(u^4\right)^{-3}$
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{\left(u^4\right)^3}$
Simplify.	$\frac{1}{u^{12}}$

TRY IT : : 6.187 Simplify: ⓐ y^{-7} ⓑ $\left(z^3\right)^{-5}$.

TRY IT : : 6.188 Simplify: ⓐ p^{-9} ⓑ $\left(q^4\right)^{-6}$.

When there is a product and an exponent we have to be careful to apply the exponent to the correct quantity. According to the Order of Operations, we simplify expressions in parentheses before applying exponents. We'll see how this works in the next example.

EXAMPLE 6.95

Simplify: ⓐ $5y^{-1}$ ⓑ $(5y)^{-1}$ ⓒ $(-5y)^{-1}$.

Solution

ⓐ

	$5y^{-1}$
Notice the exponent applies to just the base y. Take the reciprocal of y and change the sign of the exponent.	$5 \cdot \frac{1}{y^1}$
Simplify.	$\frac{5}{y}$

ⓑ

	$(5y)^{-1}$
Here the parentheses make the exponent apply to the base $5y$. Take the reciprocal of $5y$ and change the sign of the exponent.	$\frac{1}{(5y)^1}$
Simplify.	$\frac{1}{5y}$

ⓒ

	$(-5y)^{-1}$
The base here is $-5y$. Take the reciprocal of $-5y$ and change the sign of the exponent.	$\frac{1}{(-5y)^1}$
Simplify.	$\frac{1}{-5y}$
Use $\frac{a}{-b} = -\frac{a}{b}$.	$-\frac{1}{5y}$

TRY IT : : 6.189 Simplify: ⓐ $8p^{-1}$ ⓑ $(8p)^{-1}$ ⓒ $(-8p)^{-1}$.

TRY IT : : 6.190 Simplify: ⓐ $11q^{-1}$ ⓑ $(11q)^{-1}$ $-(11q)^{-1}$ ⓒ $(-11q)^{-1}$.

With negative exponents, the Quotient Rule needs only one form $\frac{a^m}{a^n} = a^{m-n}$, for $a \neq 0$. When the exponent in the denominator is larger than the exponent in the numerator, the exponent of the quotient will be negative.

Simplify Expressions with Integer Exponents

All of the exponent properties we developed earlier in the chapter with whole number exponents apply to integer exponents, too. We restate them here for reference.

Summary of Exponent Properties

If a and b are real numbers, and m and n are integers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{a^n}$	$=$	$a^{m-n}, a \neq 0$
Zero Exponent Property	a^0	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, \quad b \neq 0$
Properties of Negative Exponents	a^{-n}	$=$	$\frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n}$	$=$	$\left(\frac{b}{a}\right)^n$

EXAMPLE 6.96

Simplify: ⓐ $x^{-4} \cdot x^6$ ⓑ $y^{-6} \cdot y^4$ ⓒ $z^{-5} \cdot z^{-3}$.

Solution

ⓐ

	$x^{-4} \cdot x^6$
Use the Product Property, $a^m \cdot a^n = a^{m+n}$.	x^{-4+6}
Simplify.	x^2

ⓑ

	$y^{-6} \cdot y^{4}$
Notice the same bases, so add the exponents.	y^{-6+4}
Simplify.	y^{-2}
Use the definition of a ne ative exponent, $a^{-n} = \frac{1}{a^{n}}$.	$\frac{1}{y^{2}}$

ⓒ

	$z^{-5} \cdot z^{-3}$
Add the exponents, since the bases are the same.	z^{-5-3}
Simplify.	z^{-8}
Take the reciprocal and change the sign of the exponent, using the definition of a ne ative exponent.	$\frac{1}{z^{8}}$

TRY IT : : 6.191 Simplify: ⓐ $x^{-3} \cdot x^{7}$ ⓑ $y^{-7} \cdot y^{2}$ ⓒ $z^{-4} \cdot z^{-5}$.

TRY IT : : 6.192 Simplify: ⓐ $a^{-1} \cdot a^{6}$ ⓑ $b^{-8} \cdot b^{4}$ ⓒ $c^{-8} \cdot c^{-7}$.

In the next two examples, we'll start by using the Commutative Property to group the same variables together. This makes it easier to identify the like bases before using the Product Property.

EXAMPLE 6.97

Simplify: $\left(m^{4} n^{-3}\right)\left(m^{-5} n^{-2}\right)$.

Solution

	$\left(m^{4} n^{-3}\right)\left(m^{-5} n^{-2}\right)$
Use the Commutative Property to get like bases together.	$m^{4} m^{-5} \cdot n^{-2} n^{-3}$
Add the exponents for each base.	$m^{-1} \cdot n^{-5}$
Take reciprocals and change the signs of the exponents.	$\frac{1}{m^{1}} \cdot \frac{1}{n^{5}}$
Simplify.	$\frac{1}{mn^{5}}$

TRY IT : : 6.193 Simplify: $\left(p^{6} q^{-2}\right)\left(p^{-9} q^{-1}\right)$.

TRY IT : : 6.194 Simplify: $\left(r^{5} s^{-3}\right)\left(r^{-7} s^{-5}\right)$.

If the monomials have numerical coefficients, we multiply the coefficients, just like we did earlier.

EXAMPLE 6.98

Simplify: $\left(2x^{-6} y^{8}\right)\left(-5x^{5} y^{-3}\right)$.

Solution

	$(2x^{-6}y^{8})(-5x^{5}y^{-3})$
Rewrite with the like bases together.	$2(-5)\cdot(x^{-6}x^{5})\cdot(y^{8}y^{-3})$
Multiply the coefficients and add he exponents of each variable.	$-10\cdot x^{-1}\cdot y^{5}$
Use the definition of a ne ative exponent, $a^{-n}=\frac{1}{a^{n}}$.	$-10\cdot\frac{1}{x^{1}}\cdot y^{5}$
Simplify.	$\frac{-10y^{5}}{x}$

TRY IT : : 6.195 Simplify: $(3u^{-5}v^{7})(-4u^{4}v^{-2})$.

TRY IT : : 6.196 Simplify: $(-6c^{-6}d^{4})(-5c^{-2}d^{-1})$.

In the next two examples, we'll use the Power Property and the Product to a Power Property.

EXAMPLE 6.99

Simplify: $(6k^{3})^{-2}$.

Solution

	$(6k^{3})^{-2}$
Use the Product to a Power Property, $(ab)^{m}=a^{m}b^{m}$.	$(6)^{-2}(k^{3})^{-2}$
Use the Power Property, $(a^{m})^{n}=a^{m\cdot n}$.	$6^{-2}k^{-6}$
Use the Definition of a egative Exponent, $a^{-n}=\frac{1}{a^{n}}$.	$\frac{1}{6^{2}}\cdot\frac{1}{k^{6}}$
Simplify.	$\frac{1}{36k^{6}}$

TRY IT : : 6.197 Simplify: $(-4x^{4})^{-2}$.

TRY IT : : 6.198 Simplify: $(2b^{3})^{-4}$.

EXAMPLE 6.100

Simplify: $(5x^{-3})^{2}$.

Solution

	$\left(5x^{-3}\right)^2$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$5^2\left(x^{-3}\right)^2$
Simplify 5^2 and multiply the exponents of x using the Power Property, $(a^m)^n = a^{m \cdot n}$.	$25 \cdot x^{-6}$
Rewrite x^{-6} by using the Definition of a egative Exponent, $a^{-n} = \frac{1}{a^n}$.	$25 \cdot \frac{1}{x^6}$
Simplify.	$\frac{25}{x^6}$

TRY IT : : 6.199 Simplify: $\left(8a^{-4}\right)^2$.

TRY IT : : 6.200 Simplify: $\left(2c^{-4}\right)^3$.

To simplify a fraction, we use the Quotient Property and subtract the exponents.

EXAMPLE 6.101

Simplify: $\frac{r^5}{r^{-4}}$.

Solution

	$\frac{r^5}{r^{-4}}$
Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$.	$r^{5-(-4)}$
Simplify.	r^9

TRY IT : : 6.201 Simplify: $\frac{x^8}{x^{-3}}$.

TRY IT : : 6.202 Simplify: $\frac{y^8}{y^{-6}}$.

Convert from Decimal Notation to Scientific Notation

Remember working with place value for whole numbers and decimals? Our number system is based on powers of 10. We use tens, hundreds, thousands, and so on. Our decimal numbers are also based on powers of tens—tenths, hundredths, thousandths, and so on. Consider the numbers 4,000 and 0.004. We know that 4,000 means $4 \times 1{,}000$ and 0.004 means $4 \times \frac{1}{1{,}000}$.

If we write the 1000 as a power of ten in exponential form, we can rewrite these numbers in this way:

$4{,}000$	0.004
$4 \times 1{,}000$	$4 \times \dfrac{1}{1{,}000}$
4×10^3	$4 \times \dfrac{1}{10^3}$
	4×10^{-3}

When a number is written as a product of two numbers, where the first factor is a number greater than or equal to one but less than 10, and the second factor is a power of 10 written in exponential form, it is said to be in *scientific notation.*

Scientific Notation

A number is expressed in **scientific notation** when it is of the form

$$a \times 10^n \text{ where } 1 \leq a < 10 \text{ and } n \text{ is an integer}$$

It is customary in scientific notation to use as the $\times$ multiplication sign, even though we avoid using this sign elsewhere in algebra.

If we look at what happened to the decimal point, we can see a method to easily convert from decimal notation to scientific notation.

$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
$4000. = 4 \times 10^3$	$0.004 = 4 \times 10^{-3}$
Moved the decimal point 3 places to the left.	Moved the decimal point 3 places to the right.

In both cases, the decimal was moved 3 places to get the first factor between 1 and 10.

The power of 10 is positive when the number is larger than 1:	$4{,}000 = 4 \times 10^3$
The power of 10 is negative when the number is between 0 and 1:	$0.004 = 4 \times 10^{-3}$

EXAMPLE 6.102 HOW TO CONVERT FROM DECIMAL NOTATION TO SCIENTIFIC NOTATION

Write in scientific notation: 37,000.

Solution

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.	Remember, there is a decimal at the end of 37,000. Move the decimal after the 3. 3.700 is between 1 and 10.	37,000.
Step 2. Count the number of decimal places, *n*, that the decimal point was moved.	The decimal point was moved 4 places to the left.	37000.
Step 3. Write the number as a product with a power of 10. If the original number is: Greater than 1, the power of 10 will be 10^n. Between 0 and 1, the power of 10 will be 10^{-n}.	37,000 is greater than 1 so the power of 10 will have exponent 4.	3.7×10^4
Step 4. Check.	Check to see if your answer makes sense.	10^4 is 10,000 and 10,000 times 3.7 will be 37,000. $37{,}000 = 3.7 \times 10^4$

TRY IT : : 6.203 Write in scientific notation: $96{,}000.$

TRY IT : : 6.204 Write in scientific notation: $48{,}300$.

HOW TO : : CONVERT FROM DECIMAL NOTATION TO SCIENTIFIC NOTATION

Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

Step 2. Count the number of decimal places, *n*, that the decimal point was moved.

Step 3. Write the number as a product with a power of 10.
If the original number is:

- greater than 1, the power of 10 will be 10^n.
- between 0 and 1, the power of 10 will be 10^{-n}.

Step 4. Check.

EXAMPLE 6.103

Write in scientific notation: 0.0052.

Solution

The original number, 0.0052 , is between 0 and 1 so we will have a negative power of 10.

	0.0052
Move the decimal point to get 5.2, a number between 1 and 10.	0.0052
Count the number of decimal places the point was moved.	3 places
Write as a product with a power of 10.	5.2×10^{-3}
Check.	

$$5.2 \times 10^{-3}$$
$$5.2 \times \frac{1}{10^3}$$
$$5.2 \times \frac{1}{1000}$$
$$5.2 \times 0.001$$
$$0.0052$$

$0.0052 = 5.2 \times 10^{-3}$

TRY IT : : 6.205 Write in scientific notation: 0.0078.

TRY IT : : 6.206 Write in scientific notation: 0.0129.

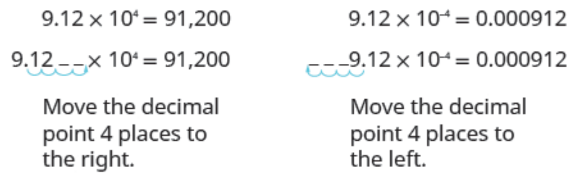
Convert Scientific Notation to Decimal Form

How can we convert from scientific notation to decimal form? Let's look at two numbers written in scientific notation and see.

$$\begin{array}{cc} 9.12 \times 10^4 & 9.12 \times 10^{-4} \\ 9.12 \times 10{,}000 & 9.12 \times 0.0001 \\ 91{,}200 & 0.000912 \end{array}$$

If we look at the location of the decimal point, we can see an easy method to convert a number from scientific notation to decimal form.

$$9.12 \times 10^4 = 91{,}200 \qquad 9.12 \times 10^{-4} = 0.000912$$



In both cases the decimal point moved 4 places. When the exponent was positive, the decimal moved to the right. When the exponent was negative, the decimal point moved to the left.

EXAMPLE 6.104 HOW TO CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM

Convert to decimal form: 6.2×10^3.

Solution

Step 1. Determine the exponent, *n*, on the factor 10.	The exponent is 3.	6.2×10^3
Step 2. Move the decimal *n* places, adding zeros if needed. If the exponent is positive, move the decimal point *n* places to the right. If the exponent is negative, move the decimal point $\lvert n \rvert$ places to the left.	The exponent is positive, so move the decimal point 3 places to the right. We need to add 2 zeros as placeholders.	6.200. 6,200
Step 3. Check to see if your answer makes sense.		10^3 is 1000 and 1000 times 6.2 will be 6,200. $6.2 \times 10^3 = 6{,}200$

TRY IT : : 6.207 Convert to decimal form: 1.3×10^3.

TRY IT : : 6.208 Convert to decimal form: 9.25×10^4.

The steps are summarized below.

HOW TO : : CONVERT SCIENTIFIC NOTATION TO DECIMAL FORM.

To convert scientific notation to decimal form:

Step 1. Determine the exponent, n, on the factor 10.

Step 2. Move the decimal n places, adding zeros if needed.

- If the exponent is positive, move the decimal point n places to the right.
- If the exponent is negative, move the decimal point $|n|$ places to the left.

Step 3. Check.

EXAMPLE 6.105

Convert to decimal form: 8.9×10^{-2}.

Solution

	8.9×10^{-2}
Determine the exponent, *n*, on the factor 10.	The exponent is −2.
Since the exponent is negative, move the decimal point 2 places to the left.	8.9
Add zeros as needed for placeholders.	$8.9 \times 10^{-2} = 0.089$

TRY IT : : 6.209 Convert to decimal form: 1.2×10^{-4}.

TRY IT : : 6.210 Convert to decimal form: 7.5×10^{-2}.

Multiply and Divide Using Scientific Notation

Astronomers use very large numbers to describe distances in the universe and ages of stars and planets. Chemists use very small numbers to describe the size of an atom or the charge on an electron. When scientists perform calculations with very large or very small numbers, they use scientific notation. Scientific notation provides a way for the calculations to be done without writing a lot of zeros. We will see how the Properties of Exponents are used to multiply and divide numbers in scientific notation.

EXAMPLE 6.106

Multiply. Write answers in decimal form: $(4 \times 10^5)(2 \times 10^{-7})$.

Solution

	$(4 \times 10^5)(2 \times 10^{-7})$
Use the Commutative Property to rearrange the factors.	$4 \cdot 2 \cdot 10^5 \cdot 10^{-7}$
Multiply.	8×10^{-2}
Change to decimal form by moving the decimal two places left.	0.08

TRY IT : : 6.211 Multiply $(3 \times 10^6)(2 \times 10^{-8})$. Write answers in decimal form.

TRY IT : : 6.212 Multiply $(3 \times 10^{-2})(3 \times 10^{-1})$. Write answers in decimal form.

EXAMPLE 6.107

Divide. Write answers in decimal form: $\dfrac{9 \times 10^3}{3 \times 10^{-2}}$.

Solution

	$\dfrac{9 \times 10^{3}}{3 \times 10^{-2}}$
Separate the factors, rewriting as the product of two fractions.	$\dfrac{9}{3} \times \dfrac{10^{3}}{10^{-2}}$
Divide.	3×10^{5}
Change to decimal form by moving the decimal fi e places right.	300,000

TRY IT : : 6.213 Divide $\dfrac{8 \times 10^{4}}{2 \times 10^{-1}}$. Write answers in decimal form.

TRY IT : : 6.214 Divide $\dfrac{8 \times 10^{2}}{4 \times 10^{-2}}$. Write answers in decimal form.

MEDIA : :

Access these online resources for additional instruction and practice with integer exponents and scientific notation:

- **Negative Exponents (https://openstax.org/l/25Negexponents)**
- **Scientific Notation (https://openstax.org/l/25Scientnot1)**
- **Scientific Notation 2 (https://openstax.org/l/25Scientnot2)**

6.7 EXERCISES

Practice Makes Perfect

Use the Definition of a Negative Exponent

In the following exercises, simplify.

500.
ⓐ 4^{-2}
ⓑ 10^{-3}

501.
ⓐ 3^{-4}
ⓑ 10^{-2}

502.
ⓐ 5^{-3}
ⓑ 10^{-5}

503.
ⓐ 2^{-8}
ⓑ 10^{-2}

504.
ⓐ $\frac{1}{c^{-5}}$
ⓑ $\frac{1}{3^{-2}}$

505.
ⓐ $\frac{1}{c^{-5}}$
ⓑ $\frac{1}{5^{-2}}$

506.
ⓐ $\frac{1}{q^{-10}}$
ⓑ $\frac{1}{10^{-3}}$

507.
ⓐ $\frac{1}{t^{-9}}$
ⓑ $\frac{1}{10^{-4}}$

508.
ⓐ $\left(\frac{5}{8}\right)^{-2}$
ⓑ $\left(-\frac{3m}{n}\right)^{-2}$

509.
ⓐ $\left(\frac{3}{10}\right)^{-2}$
ⓑ $\left(-\frac{2}{cd}\right)^{-3}$

510.
ⓐ $\left(\frac{4}{9}\right)^{-3}$
ⓑ $\left(-\frac{u^2}{2v}\right)^{-5}$

511.
ⓐ $\left(\frac{7}{2}\right)^{-3}$
ⓑ $\left(-\frac{3}{xy^2}\right)^{-3}$

512.
ⓐ $(-5)^{-2}$
ⓑ -5^{-2}
ⓒ $\left(-\frac{1}{5}\right)^{-2}$
ⓓ $-\left(\frac{1}{5}\right)^{-2}$

513.
ⓐ $(-7)^{-2}$
ⓑ -7^{-2}
ⓒ $\left(-\frac{1}{7}\right)^{-2}$
ⓓ $-\left(\frac{1}{7}\right)^{-2}$

514.
ⓐ -3^{-3}
ⓑ $\left(-\frac{1}{3}\right)^{-3}$
ⓒ $-\left(\frac{1}{3}\right)^{-3}$
ⓓ $(-3)^{-3}$

515.
ⓐ -5^{-3}
ⓑ $\left(-\frac{1}{5}\right)^{-3}$
ⓒ $-\left(\frac{1}{5}\right)^{-3}$
ⓓ $(-5)^{-3}$

516.
ⓐ $3 \cdot 5^{-1}$
ⓑ $(3 \cdot 5)^{-1}$

517.
ⓐ $2 \cdot 5^{-1}$
ⓑ $(2 \cdot 5)^{-1}$

518.
ⓐ $4 \cdot 5^{-2}$
ⓑ $(4 \cdot 5)^{-2}$

519.
ⓐ $3 \cdot 4^{-2}$
ⓑ $(3 \cdot 4)^{-2}$

520.
ⓐ m^{-4}
ⓑ $\left(x^3\right)^{-4}$

521.
ⓐ b^{-5}
ⓑ $\left(k^2\right)^{-5}$

522.
ⓐ p^{-10}
ⓑ $\left(q^6\right)^{-8}$

523.
ⓐ s^{-8}
ⓑ $\left(a^9\right)^{-10}$

524.
ⓐ $7n^{-1}$
ⓑ $(7n)^{-1}$
ⓒ $(-7n)^{-1}$

525.
ⓐ $6r^{-1}$
ⓑ $(6r)^{-1}$
ⓒ $(-6r)^{-1}$

526.
ⓐ $(3p)^{-2}$
ⓑ $3p^{-2}$
ⓒ $-3p^{-2}$

527.
ⓐ $(2q)^{-4}$
ⓑ $2q^{-4}$
ⓒ $-2q^{-4}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

528.
ⓐ $b^4 b^{-8}$
ⓑ $r^{-2} r^5$
ⓒ $x^{-7} x^{-3}$

529.
ⓐ $s^3 \cdot s^{-7}$
ⓑ $q^{-8} \cdot q^3$
ⓒ $y^{-2} \cdot y^{-5}$

530.
ⓐ $a^3 \cdot a^{-3}$
ⓑ $a \cdot a^3$
ⓒ $a \cdot a^{-3}$

531.
ⓐ $y^5 \cdot y^{-5}$
ⓑ $y \cdot y^5$
ⓒ $y \cdot y^{-5}$

532. $p^5 \cdot p^{-2} \cdot p^{-4}$

533. $x^4 \cdot x^{-2} \cdot x^{-3}$

534. $\left(w^4 x^{-5}\right)\left(w^{-2} x^{-4}\right)$

535. $\left(m^3 n^{-3}\right)\left(m^{-5} n^{-1}\right)$

536. $\left(uv^{-2}\right)\left(u^{-5} v^{-3}\right)$

537. $\left(pq^{-4}\right)\left(p^{-6} q^{-3}\right)$

538. $\left(-6c^{-3} d^9\right)\left(2c^4 d^{-5}\right)$

539. $\left(-2j^{-5} k^8\right)\left(7j^2 k^{-3}\right)$

540. $\left(-4r^{-2} s^{-8}\right)\left(9r^4 s^3\right)$

541. $\left(-5m^4 n^6\right)\left(8m^{-5} n^{-3}\right)$

542. $\left(5x^2\right)^{-2}$

543. $\left(4y^3\right)^{-3}$

544. $\left(3z^{-3}\right)^2$

545. $\left(2p^{-5}\right)^2$

546. $\dfrac{t^9}{t^{-3}}$

547. $\dfrac{n^5}{n^{-2}}$

548. $\dfrac{x^{-7}}{x^{-3}}$

549. $\dfrac{y^{-5}}{y^{-10}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

550. 57,000

551. 340,000

552. 8,750,000

553. 1,290,000

554. 0.026

555. 0.041

556. 0.00000871

557. 0.00000103

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

558. 5.2×10^2

559. 8.3×10^2

560. 7.5×10^6

561. 1.6×10^{10}

562. 2.5×10^{-2}

563. 3.8×10^{-2}

564. 4.13×10^{-5}

565. 1.93×10^{-5}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply. Write your answer in decimal form.

566. $\left(3 \times 10^{-5}\right)\left(3 \times 10^{9}\right)$

567. $\left(2 \times 10^{2}\right)\left(1 \times 10^{-4}\right)$

568. $\left(7.1 \times 10^{-2}\right)\left(2.4 \times 10^{-4}\right)$

569. $\left(3.5 \times 10^{-4}\right)\left(1.6 \times 10^{-2}\right)$

In the following exercises, divide. Write your answer in decimal form.

570. $\dfrac{7 \times 10^{-3}}{1 \times 10^{-7}}$

571. $\dfrac{5 \times 10^{-2}}{1 \times 10^{-10}}$

572. $\dfrac{6 \times 10^{4}}{3 \times 10^{-2}}$

573. $\dfrac{8 \times 10^{6}}{4 \times 10^{-1}}$

Everyday Math

574. The population of the United States on July 4, 2010 was almost 310,000,000. Write the number in scientific notation.

575. The population of the world on July 4, 2010 was more than 6,850,000,000. Write the number in scientific notation

576. The average width of a human hair is 0.0018 centimeters. Write the number in scientific notation.

577. The probability of winning the 2010 Megamillions lottery was about 0.0000000057. Write the number in scientific notation.

578. In 2010, the number of Facebook users each day who changed their status to 'engaged' was 2×10^4. Convert this number to decimal form.

579. At the start of 2012, the US federal budget had a deficit of more than $\$1.5 \times 10^{13}$. Convert this number to decimal form.

580. The concentration of carbon dioxide in the atmosphere is 3.9×10^{-4}. Convert this number to decimal form.

581. The width of a proton is 1×10^{-5} of the width of an atom. Convert this number to decimal form.

582. Health care costs The Centers for Medicare and Medicaid projects that consumers will spend more than $4 trillion on health care by 2017.

ⓐ Write 4 trillion in decimal notation.

ⓑ Write 4 trillion in scientific notation.

583. Coin production In 1942, the U.S. Mint produced 154,500,000 nickels. Write 154,500,000 in scientific notation.

584. Distance The distance between Earth and one of the brightest stars in the night star is 33.7 light years. One light year is about 6,000,000,000,000 (6 trillion), miles.

ⓐ Write the number of miles in one light year in scientific notation.

ⓑUse scientific notation to find the distance between Earth and the star in miles. Write the answer in scientific notation.

585. Debt At the end of fiscal year 2015 the gross United States federal government debt was estimated to be approximately $18,600,000,000,000 ($18.6 trillion), according to the Federal Budget. The population of the United States was approximately 300,000,000 people at the end of fiscal year 2015.

ⓐ Write the debt in scientific notation.

ⓑ Write the population in scientific notation.

ⓒ Find the amount of debt per person by using scientific notation to divide the debt by the population. Write the answer in scientific notation.

Writing Exercises

586.

ⓐ Explain the meaning of the exponent in the expression 2^3.

ⓑ Explain the meaning of the exponent in the expression 2^{-3}.

587. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the definition of a negative exponent.			
simplify expressions with integer exponents.			
convert from decimal notation to scientific notation.			
convert scientific notation to decimal form.			
multiply and divide using scientific notation.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

CHAPTER 6 REVIEW

KEY TERMS

binomial A binomial is a polynomial with exactly two terms.

conjugate pair A conjugate pair is two binomials of the form $(a - b), (a + b)$; the pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

degree of a constant The degree of any constant is 0.

degree of a polynomial The degree of a polynomial is the highest degree of all its terms.

degree of a term The degree of a term is the exponent of its variable.

monomial A monomial is a term of the form ax^m, where a is a constant and m is a whole number; a monomial has exactly one term.

negative exponent If n is a positive integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

polynomial A polynomial is a monomial, or two or more monomials combined by addition or subtraction.

scientific notation A number is expressed in scientific notation when it is of the form $a \times 10^n$ where $a \geq 1$ and $a < 10$ and n is an integer.

standard form A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial A trinomial is a polynomial with exactly three terms.

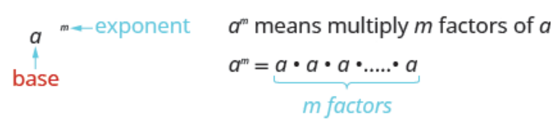
KEY CONCEPTS

6.1 Add and Subtract Polynomials

- **Monomials**
  - A monomial is a term of the form ax^m, where a is a constant and m is a whole number
- **Polynomials**
  - **polynomial**—A monomial, or two or more monomials combined by addition or subtraction is a polynomial.
  - **monomial**—A polynomial with exactly one term is called a monomial.
  - **binomial**—A polynomial with exactly two terms is called a binomial.
  - **trinomial**—A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

6.2 Use Multiplication Properties of Exponents

- **Exponential Notation**



- **Properties of Exponents**
  - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$

6.3 Multiply Polynomials

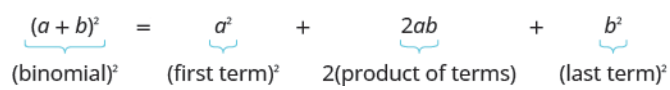
- **FOIL Method for Multiplying Two Binomials**—To multiply two binomials:
  Step 1. Multiply the **First** terms.
  Step 2. Multiply the **Outer** terms.
  Step 3. Multiply the **Inner** terms.
  Step 4. Multiply the **Last** terms.

- **Multiplying Two Binomials**—To multiply binomials, use the:
  - Distributive Property (**Example 6.34**)
  - FOIL Method (**Example 6.39**)
  - Vertical Method (**Example 6.44**)

- **Multiplying a Trinomial by a Binomial**—To multiply a trinomial by a binomial, use the:
  - Distributive Property (**Example 6.45**)
  - Vertical Method (**Example 6.46**)

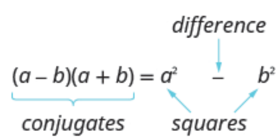
6.4 Special Products

- **Binomial Squares Pattern**
  - If a, b are real numbers,



  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a-b)^2 = a^2 - 2ab + b^2$
  - To square a binomial: square the first term, square the last term, double their product.

- **Product of Conjugates Pattern**
  - If a, b are real numbers,



  - $(a-b)(a+b) = a^2 - b^2$
  - The product is called a difference of squares.

- **To multiply conjugates:**
  - **square the first term square the last term** write it as a difference of squares

6.5 Divide Monomials

- **Quotient Property for Exponents:**
  - If a is a real number, $a \neq 0$, and m, n are whole numbers, then:
    $\frac{a^m}{a^n} = a^{m-n},\ m > n$ and $\frac{a^m}{a^n} = \frac{1}{a^{m-n}},\ n > m$

- **Zero Exponent**
  - If a is a non-zero number, then $a^0 = 1$.

- **Quotient to a Power Property for Exponents**:
    - If a and b are real numbers, $b \neq 0,$ and m is a counting number, then:
      $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
    - To raise a fraction to a power, raise the numerator and denominator to that power.

- **Summary of Exponent Properties**
    - If a, b are real numbers and m, n are whole numbers, then

Product Property	$a^m \cdot a^n$	$=$	a^{m+n}
Power Property	$(a^m)^n$	$=$	$a^{m \cdot n}$
Product to a Power	$(ab)^m$	$=$	$a^m b^m$
Quotient Property	$\frac{a^m}{b^m}$	$=$	$a^{m-n}, a \neq 0, m > n$
	$\frac{a^m}{a^n}$	$=$	$\frac{1}{a^{n-m}}, a \neq 0, n > m$
Zero Exponent Definitio	a^o	$=$	$1, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m$	$=$	$\frac{a^m}{b^m}, b \neq 0$

6.6 Divide Polynomials

- **Fraction Addition**
    - If $a, b,$ and c are numbers where $c \neq 0$, then
      $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$

- **Division of a Polynomial by a Monomial**
    - To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

6.7 Integer Exponents and Scientific Notation

- **Property of Negative Exponents**
    - If n is a positive integer and $a \neq 0$, then $\frac{1}{a^{-n}} = a^n$

- **Quotient to a Negative Exponent**
    - If a, b are real numbers, $b \neq 0$ and n is an integer , then $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

- **To convert a decimal to scientific notation:**

  Step 1. Move the decimal point so that the first factor is greater than or equal to 1 but less than 10.

  Step 2. Count the number of decimal places, n, that the decimal point was moved.

  Step 3. Write the number as a product with a power of 10. If the original number is:
    - greater than 1, the power of 10 will be 10^n
    - between 0 and 1, the power of 10 will be 10^{-n}

  Step 4. Check.

- **To convert scientific notation to decimal form:**

  Step 1. Determine the exponent, n, on the factor 10.

  Step 2. Move the decimal n places, adding zeros if needed.
    - If the exponent is positive, move the decimal point n places to the right.

- If the exponent is negative, move the decimal point $|n|$ places to the left.

Step 3. Check.

REVIEW EXERCISES

6.1 Section 6.1 Add and Subtract Polynomials

Identify Polynomials, Monomials, Binomials and Trinomials

In the following exercises, determine if each of the following polynomials is a monomial, binomial, trinomial, or other polynomial.

588.
ⓐ $11c^4 - 23c^2 + 1$
ⓑ $9p^3 + 6p^2 - p - 5$
ⓒ $\frac{3}{7}x + \frac{5}{14}$
ⓓ 10
ⓔ $2y - 12$

589.
ⓐ $a^2 - b^2$
ⓑ $24d^3$
ⓒ $x^2 + 8x - 10$
ⓓ $m^2n^2 - 2mn + 6$
ⓔ $7y^3 + y^2 - 2y - 4$

Determine the Degree of Polynomials

In the following exercises, determine the degree of each polynomial.

590.
ⓐ $3x^2 + 9x + 10$
ⓑ $14a^2bc$
ⓒ $6y + 1$
ⓓ $n^3 - 4n^2 + 2n - 8$
ⓔ -19

591.
ⓐ $5p^3 - 8p^2 + 10p - 4$
ⓑ $-20q^4$
ⓒ $x^2 + 6x + 12$
ⓓ $23r^2s^2 - 4rs + 5$
ⓔ 100

Add and Subtract Monomials

In the following exercises, add or subtract the monomials.

592. $5y^3 + 8y^3$

593. $-14k + 19k$

594. $12q - (-6q)$

595. $-9c - 18c$

596. $12x - 4y - 9x$

597. $3m^2 + 7n^2 - 3m^2$

598. $6x^2y - 4x + 8xy^2$

599. $13a + b$

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

600. $(5x^2 + 12x + 1) + (6x^2 - 8x + 3)$

601. $(9p^2 - 5p + 3) + (4p^2 - 4)$

602. $(10m^2 - 8m - 1) - (5m^2 + m - 2)$

603. $(7y^2 - 8y) - (y - 4)$

604. Subtract $(3s^2 + 10)$ from $(15s^2 - 2s + 8)$

605. Find the sum of $(a^2 + 6a + 9)$ and $(5a^3 - 7)$

Evaluate a Polynomial for a Given Value of the Variable

In the following exercises, evaluate each polynomial for the given value.

606. Evaluate $3y^2 - y + 1$ when:

ⓐ $y = 5$ ⓑ $y = -1$

ⓒ $y = 0$

607. Evaluate $10 - 12x$ when:

ⓐ $x = 3$ ⓑ $x = 0$

ⓒ $x = -1$

608. Randee drops a stone off the 200 foot high cliff into the ocean. The polynomial $-16t^2 + 200$ gives the height of a stone t seconds after it is dropped from the cliff. Find the height after $t = 3$ seconds.

609. A manufacturer of stereo sound speakers has found that the revenue received from selling the speakers at a cost of *p* dollars each is given by the polynomial $-4p^2 + 460p$. Find the revenue received when $p = 75$ dollars.

6.2 Section 6.2 Use Multiplication Properties of Exponents

Simplify Expressions with Exponents

In the following exercises, simplify.

610. 10^4

611. 17^1

612. $\left(\frac{2}{9}\right)^2$

613. $(0.5)^3$

614. $(-2)^6$

615. -2^6

Simplify Expressions Using the Product Property for Exponents

In the following exercises, simplify each expression.

616. $x^4 \cdot x^3$

617. $p^{15} \cdot p^{16}$

618. $4^{10} \cdot 4^6$

619. $8 \cdot 8^5$

620. $n \cdot n^2 \cdot n^4$

621. $y^c \cdot y^3$

Simplify Expressions Using the Power Property for Exponents

In the following exercises, simplify each expression.

622. $(m^3)^5$

623. $(5^3)^2$

624. $\left(y^4\right)^x$

625. $(3^r)^s$

Simplify Expressions Using the Product to a Power Property

In the following exercises, simplify each expression.

626. $(4a)^2$

627. $(-5y)^3$

628. $(2mn)^5$

629. $(10xyz)^3$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify each expression.

630. $\left(p^2\right)^5 \cdot \left(p^3\right)^6$

631. $(4a^3b^2)^3$

632. $(5x)^2(7x)$

633. $(2q^3)^4(3q)^2$

634. $\left(\frac{1}{3}x^2\right)^2\left(\frac{1}{2}x\right)^3$

635. $\left(\frac{2}{5}m^2n\right)^3$

Multiply Monomials

In the following exercises 8, multiply the monomials.

636. $\left(-15x^2\right)\left(6x^4\right)$

637. $(-9n^7)(-16n)$

638. $(7p^5q^3)(8pq^9)$

639. $\left(\frac{5}{9}ab^2\right)\left(27ab^3\right)$

6.3 Section 6.3 Multiply Polynomials

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

640. $7(a+9)$

641. $-4(y+13)$

642. $-5(r-2)$

643. $p(p+3)$

644. $-m(m+15)$

645. $-6u(2u+7)$

646. $9(b^2+6b+8)$

647. $3q^2(q^2-7q+6)$ 3

648. $(5z-1)z$

649. $(b-4)\cdot 11$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using: ⓐ the Distributive Property, ⓑ the FOIL method, ⓒ the Vertical Method.

650. $(x-4)(x+10)$

651. $(6y-7)(2y-5)$

In the following exercises, multiply the binomials. Use any method.

652. $(x+3)(x+9)$

653. $(y-4)(y-8)$

654. $(p-7)(p+4)$

655. $(q+16)(q-3)$

656. $(5m-8)(12m+1)$

657. $(u^2+6)(u^2-5)$

658. $(9x-y)(6x-5)$

659. $(8mn+3)(2mn-1)$

Multiply a Trinomial by a Binomial

In the following exercises, multiply using ⓐ the Distributive Property, ⓑ the Vertical Method.

660. $(n+1)\left(n^2+5n-2\right)$

661. $(3x-4)\left(6x^2+x-10\right)$

In the following exercises, multiply. Use either method.

662. $(y-2)(y^2-8y+9)$

663. $(7m+1)(m^2-10m-3)$

6.4 Section 6.4 Special Products

Square a Binomial Using the Binomial Squares Pattern

In the following exercises, square each binomial using the Binomial Squares Pattern.

664. $(c+11)^2$

665. $(q-15)^2$

666. $\left(x+\frac{1}{3}\right)^2$

667. $(8u+1)^2$

668. $(3n^3-2)^2$

669. $(4a-3b)^2$

Multiply Conjugates Using the Product of Conjugates Pattern

In the following exercises, multiply each pair of conjugates using the Product of Conjugates Pattern.

670. $(s-7)(s+7)$

671. $\left(y+\frac{2}{5}\right)\left(y-\frac{2}{5}\right)$

672. $(12c+13)(12c-13)$

673. $(6-r)(6+r)$

674. $\left(u+\frac{3}{4}v\right)\left(u-\frac{3}{4}v\right)$

675. $(5p^4-4q^3)(5p^4+4q^3)$

Recognize and Use the Appropriate Special Product Pattern

In the following exercises, find each product.

676. $(3m+10)^2$

677. $(6a+11)(6a-11)$

678. $(5x+y)(x-5y)$

679. $(c^4+9d)^2$

680. $(p^5+q^5)(p^5-q^5)$

681. $(a^2+4b)(4a-b^2)$

6.5 Section 6.5 Divide Monomials

Simplify Expressions Using the Quotient Property for Exponents

In the following exercises, simplify.

682. $\frac{u^{24}}{u^6}$

683. $\frac{10^{25}}{10^5}$

684. $\frac{3^4}{3^6}$

685. $\frac{v^{12}}{v^{48}}$

686. $\frac{x}{x^5}$

687. $\frac{5}{5^8}$

Simplify Expressions with Zero Exponents

In the following exercises, simplify.

688. 75^0

689. x^0

690. -12^0

691. $\left(-12^0\right)(-12)^0$

692. $25x^0$

693. $(25x)^0$

694. $19n^0-25m^0$

695. $(19n)^0-(25m)^0$

Simplify Expressions Using the Quotient to a Power Property

In the following exercises, simplify.

696. $\left(\frac{2}{5}\right)^3$

697. $\left(\frac{m}{3}\right)^4$

698. $\left(\frac{r}{s}\right)^8$

699. $\left(\frac{x}{2y}\right)^6$

Simplify Expressions by Applying Several Properties

In the following exercises, simplify.

700. $\frac{\left(x^3\right)^5}{x^9}$

701. $\frac{n^{10}}{\left(n^5\right)^2}$

702. $\left(\frac{q^6}{q^8}\right)^3$

703. $\left(\frac{r^8}{r^3}\right)^4$

704. $\left(\frac{c^2}{d^5}\right)^9$

705. $\left(\frac{3x^4}{2y^2}\right)^5$

706. $\left(\frac{v^3 v^9}{v^6}\right)^4$

707. $\frac{\left(3n^2\right)^4\left(-5n^4\right)^3}{\left(-2n^5\right)^2}$

Divide Monomials

In the following exercises, divide the monomials.

708. $-65y^{14} \div 5y^2$

709. $\frac{64a^5 b^9}{-16a^{10} b^3}$

710. $\frac{144x^{15} y^8 z^3}{18x^{10} y^2 z^{12}}$

711. $\frac{\left(8p^6 q^2\right)\left(9p^3 q^5\right)}{16p^8 q^7}$

6.6 Section 6.6 Divide Polynomials

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial.

712. $\frac{42z^2 - 18z}{6}$

713. $\left(35x^2 - 75x\right) \div 5x$

714. $\frac{81n^4 + 105n^2}{-3}$

715. $\frac{550p^6 - 300p^4}{10p^3}$

716. $\left(63xy^3 + 56x^2 y^4\right) \div (7xy)$

717. $\frac{96a^5 b^2 - 48a^4 b^3 - 56a^2 b^4}{8ab^2}$

718. $\frac{57m^2 - 12m + 1}{-3m}$

719. $\frac{105y^5 + 50y^3 - 5y}{5y^3}$

Divide a Polynomial by a Binomial

In the following exercises, divide each polynomial by the binomial.

720. $\left(k^2 - 2k - 99\right) \div (k + 9)$

721. $\left(v^2 - 16v + 64\right) \div (v - 8)$

722. $\left(3x^2 - 8x - 35\right) \div (x - 5)$

723. $\left(n^2 - 3n - 14\right) \div (n + 3)$

724. $\left(4m^3 + m - 5\right) \div (m - 1)$

725. $\left(u^3 - 8\right) \div (u - 2)$

6.7 Section 6.7 Integer Exponents and Scientific Notation

Use the Definition of a Negative Exponent

In the following exercises, simplify.

726. 9^{-2}

727. $(-5)^{-3}$

728. $3 \cdot 4^{-3}$

729. $(6u)^{-3}$

730. $\left(\frac{2}{5}\right)^{-1}$

731. $\left(\frac{3}{4}\right)^{-2}$

Simplify Expressions with Integer Exponents

In the following exercises, simplify.

732. $p^{-2} \cdot p^{8}$

733. $q^{-6} \cdot q^{-5}$

734. $\left(c^{-2} d\right)\left(c^{-3} d^{-2}\right)$

735. $\left(y^{8}\right)^{-1}$

736. $\left(q^{-4}\right)^{-3}$

737. $\frac{a^{8}}{a^{12}}$

738. $\frac{n^{5}}{n^{-4}}$

739. $\frac{r^{-2}}{r^{-3}}$

Convert from Decimal Notation to Scientific Notation

In the following exercises, write each number in scientific notation.

740. 8,500,000

741. 0.00429

742. The thickness of a dime is about 0.053 inches.

743. In 2015, the population of the world was about 7,200,000,000 people.

Convert Scientific Notation to Decimal Form

In the following exercises, convert each number to decimal form.

744. 3.8×10^{5}

745. 1.5×10^{10}

746. 9.1×10^{-7}

747. 5.5×10^{-1}

Multiply and Divide Using Scientific Notation

In the following exercises, multiply and write your answer in decimal form.

748. $\left(2 \times 10^{5}\right)\left(4 \times 10^{-3}\right)$

749. $\left(3.5 \times 10^{-2}\right)\left(6.2 \times 10^{-1}\right)$

In the following exercises, divide and write your answer in decimal form.

750. $\frac{8 \times 10^{5}}{4 \times 10^{-1}}$

751. $\frac{9 \times 10^{-5}}{3 \times 10^{2}}$

PRACTICE TEST

752. For the polynomial $10x^4 + 9y^2 - 1$

ⓐ Is it a monomial, binomial, or trinomial?

ⓑ What is its degree?

In the following exercises, simplify each expression.

753. $\left(12a^2 - 7a + 4\right) + \left(3a^2 + 8a - 10\right)$

754. $\left(9p^2 - 5p + 1\right) - \left(2p^2 - 6\right)$

755. $\left(-\frac{2}{5}\right)^3$

756. $u \cdot u^4$

757. $\left(4a^3 b^5\right)^2$

758. $\left(-9r^4 s^5\right)\left(4rs^7\right)$

759. $3k\left(k^2 - 7k + 13\right)$

760. $(m + 6)(m + 12)$

761. $(v - 9)(9v - 5)$

762. $(4c - 11)(3c - 8)$

763. $(n - 6)\left(n^2 - 5n + 4\right)$

764. $(2x - 15y)(5x + 7y)$

765. $(7p - 5)(7p + 5)$

766. $(9v - 2)^2$

767. $\frac{3^8}{3^{10}}$

768. $\left(\frac{m^4 \cdot m}{m^3}\right)^6$

769. $\left(87x^{15} y^3 z^{22}\right)^0$

770. $\frac{80c^8 d^2}{16cd^{10}}$

771. $\frac{12x^2 + 42x - 6}{2x}$

772. $\left(70xy^4 + 95x^3 y\right) \div 5xy$

773. $\frac{64x^3 - 1}{4x - 1}$

774. $\left(y^2 - 5y - 18\right) \div (y + 3)$

775. 5^{-2}

776. $(4m)^{-3}$

777. $q^{-4} \cdot q^{-5}$

778. $\frac{n^{-2}}{n^{-10}}$

779. Convert 83,000,000 to scientific notation.

780. Convert 6.91×10^{-5} to decimal form.

In the following exercises, simplify, and write your answer in decimal form.

781. $\left(3.4 \times 10^9\right)\left(2.2 \times 10^{-5}\right)$

782. $\frac{8.4 \times 10^{-3}}{4 \times 10^3}$

783. A helicopter flying at an altitude of 1000 feet drops a rescue package. The polynomial $-16t^2 + 1000$ gives the height of the package t seconds a after it was dropped. Find the height when $t = 6$ seconds.

7

FACTORING

Figure 7.1 The Sydney Harbor Bridge is one of Australia's most photographed landmarks. It is the world's largest steel arch bridge with the top of the bridge standing 134 meters above the harbor. Can you see why it is known by the locals as the "Coathanger"?

Chapter Outline

Introduction

Quadratic expressions may be used to model physical properties of a large bridge, the trajectory of a baseball or rocket, and revenue and profit of a business. By factoring these expressions, specific characteristics of the model can be identified. In this chapter, you will explore the process of factoring expressions and see how factoring is used to solve certain types of equations.

7.1 Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

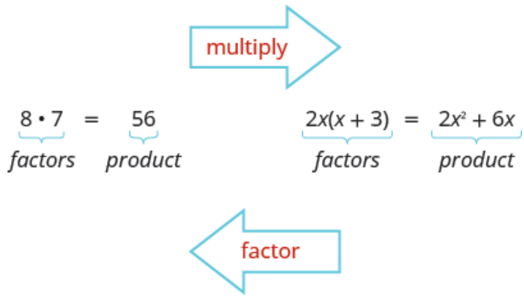
Be Prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review **Example 1.7**.
2. Find the least common multiple of 18 and 24.
   If you missed this problem, review **Example 1.10**.
3. Simplify $-3(6a + 11)$.
   If you missed this problem, review **Example 1.135**.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will be reversing this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

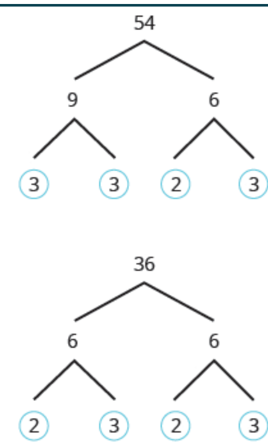
The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

First we'll find the GCF of two numbers.

EXAMPLE 7.1 HOW TO FIND THE GREATEST COMMON FACTOR OF TWO OR MORE EXPRESSIONS

Find the GCF of 54 and 36.

Solution

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.	Factor 54 and 36.	
Step 2. In each column, circle the common factors.	Circle the 2, 3, and 3 that are shared by both numbers.	36 = 2 • 2 • 3 • 3 18 = 2 • 3 • 3 • 3
Step 3. Bring down the common factors that all expressions share.	Bring down the 2, 3, and 3, and then multiply.	GCF = 2 • 3 • 3
Step 4. Multiply the factors.		GCF = 18 The GCF of 54 and 36 is 18.

Notice that, because the GCF is a factor of both numbers, 54 and 36 can be written as multiples of 18.

$$54 = 18 \cdot 3$$
$$36 = 18 \cdot 2$$

TRY IT : : 7.1 Find the GCF of 48 and 80.

TRY IT : : 7.2 Find the GCF of 18 and 40.

We summarize the steps we use to find the GCF below.

HOW TO : : FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

In the first example, the GCF was a constant. In the next two examples, we will get variables in the greatest common factor.

EXAMPLE 7.2

Find the greatest common factor of $27x^3$ and $18x^4$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$27x^3 = 3 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x$ $18x^4 = 2 \cdot 3 \cdot 3 \cdot x \cdot x \cdot x \cdot x$
Bring down the common factors.	$\text{GCF} = 3 \cdot 3 \cdot x \cdot x \cdot x$
Multiply the factors.	$\text{GCF} = 9x^3$
	The GCF of $27x^3$ and $18x^4$ is $9x^3$.

TRY IT : : 7.3 Find the GCF: $12x^2$, $18x^3$.

TRY IT : : 7.4 Find the GCF: $16y^2$, $24y^3$.

EXAMPLE 7.3

Find the GCF of $4x^2y$, $6xy^3$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$4x^2y = 2 \cdot 2 \cdot x \cdot x \cdot y$ $6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$
Bring down the common factors.	$\text{GCF} = 2 \cdot x \cdot y$
Multiply the factors.	$\text{GCF} = 2xy$

The GCF of $4x^2y$ and $6xy^3$ is 2xy .

TRY IT : : 7.5 Find the GCF: $6ab^4,\ 8a^2b$.

TRY IT : : 7.6 Find the GCF: $9m^5n^2,\ 12m^3n$.

EXAMPLE 7.4

Find the GCF of: $21x^3,\ 9x^2,\ 15x$.

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$
Bring down the common factors.	GCF = $3 \cdot x$
Multiply the factors.	GCF = $3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

TRY IT : : 7.7 Find the greatest common factor: $25m^4,\ 35m^3,\ 20m^2$.

TRY IT : : 7.8 Find the greatest common factor: $14x^3,\ 70x^2,\ 105x$.

Factor the Greatest Common Factor from a Polynomial

Just like in arithmetic, where it is sometimes useful to represent a number in factored form (for example, 12 as $2 \cdot 6 \text{ or } 3 \cdot 4$), in algebra, it can be useful to represent a polynomial in factored form. One way to do this is by finding the GCF of all the terms. Remember, we multiply a polynomial by a monomial as follows:

$$\begin{array}{cl} 2(x+7) & \text{factors} \\ 2 \cdot x + 2 \cdot 7 & \\ 2x + 14 & \text{product} \end{array}$$

Now we will start with a product, like $2x + 14$, and end with its factors, $2(x + 7)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If $a,\ b,\ c$ are real numbers, then

$$a(b + c) = ab + ac \qquad \text{and} \qquad ab + ac = a(b + c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 7.5 HOW TO FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL

Factor: $4x + 12$.

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $4x$ and 12.	$4x = 2 \cdot 2 \cdot \quad \cdot x$ $12 = 2 \cdot 2 \cdot 3$ $\text{GCF} = 2 \cdot 2$ $\text{GCF} = 4$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $4x$ and 12 as products of their GCF, 4. $4x = 4 \cdot x$ $12 = 4 \cdot 3$	$4x + 12$ $4 \cdot x + 4 \cdot 3$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4(x + 3)$
Step 4. Check by multiplying the factors.		$4(x + 3)$ $4 \cdot x + 4 \cdot 3$ $4x + 12$ ✓

TRY IT : : 7.9 Factor: $6a + 24$.

TRY IT : : 7.10 Factor: $2b + 14$.

HOW TO : : FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the "reverse" Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use "factor" as both a noun and a verb.

Noun	7 is a factor of 14
Verb	factor 3 from $3a + 3$

EXAMPLE 7.6

Factor: $5a + 5$.

Solution

Find the GCF of $5a$ and 5.	$5a = 5 \cdot a$ $5 = 5$ $\text{GCF} = 5$
	$5a + 5$
Rewrite each term as a product using the GCF.	$5 \cdot a + 5 \cdot 1$
Use the Distributive Property "in reverse" to factor the GCF.	$5(a + 1)$
Check by mulitplying the factors to get the orginal polynomial.	
$5(a + 1)$	
$5 \cdot a + 5 \cdot 1$	
$5a + 5$ ✓	

TRY IT : : 7.11 Factor: $14x + 14$.

TRY IT : : 7.12 Factor: $12p + 12$.

The expressions in the next example have several factors in common. Remember to write the GCF as the product of all the common factors.

EXAMPLE 7.7

Factor: $12x - 60$.

Solution

Find the GCF of $12x$ and 60.	$12x = 2 \cdot 2 \cdot 3 \cdot x$ $60 = 2 \cdot 2 \cdot 3 \cdot 5$ $\text{GCF} = 2 \cdot 2 \cdot 3$ $\text{GCF} = 12$
	$12x - 60$
Rewrite each term as a product using the GCF.	$12 \cdot x - 12 \cdot 5$
Factor the GCF.	$12(x - 5)$
Check by mulitplying the factors.	
$12(x - 5)$	
$12 \cdot x - 12 \cdot 5$	
$12x - 60$ ✓	

TRY IT : : 7.13 Factor: $18u - 36$.

TRY IT : : 7.14 Factor: $30y - 60$.

Now we'll factor the greatest common factor from a trinomial. We start by finding the GCF of all three terms.

EXAMPLE 7.8

Factor: $4y^2 + 24y + 28$.

Solution

We start by finding the GCF of all three terms.

Find the GCF of $4y^2$, $24y$ and 28.	$4y^2 = 2 \cdot 2 \cdot y \cdot y$ $24y = 2 \cdot 2 \cdot 2 \cdot 3 \cdot y$ $28 = 2 \cdot 2 \cdot 7$ GCF $= 2 \cdot 2$ GCF $= 4$
	$4y^2 + 24y + 28$
Rewrite each term as a product using the GCF.	$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$
Factor the GCF.	$4(y^2 + 6y + 7)$
Check by mulitplying.	
$4\left(y^2 + 6y + 7\right)$	
$4 \cdot y^2 + 4 \cdot 6y + 4 \cdot 7$	
$4y^2 + 24y + 28$ ✓	

TRY IT : : 7.15 Factor: $5x^2 - 25x + 15$.

TRY IT : : 7.16 Factor: $3y^2 - 12y + 27$.

EXAMPLE 7.9

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ GCF $= 5 \cdot x \cdot x$ GCF $= 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$

Factor the GCF.	$5x^2(x-5)$
Check.	
$5x^2(x-5)$	
$5x^2 \cdot x - 5x^2 \cdot 5$	
$5x^3 - 25x^2$ ✓	

TRY IT : : 7.17 Factor: $2x^3 + 12x^2$.

TRY IT : : 7.18 Factor: $6y^3 - 15y^2$.

EXAMPLE 7.10

Factor: $21x^3 - 9x^2 + 15x$.

Solution

In a previous example we found the GCF of $21x^3, 9x^2, 15x$ to be $3x$.

	$21x^3 - 9x^2 + 15x$
Rewrite each term using the GCF, 3*x*.	$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$
Factor the GCF.	$3x(7x^2 - 3x + 5)$
Check.	
$3x(7x^2 - 3x + 5)$	
$3x \cdot 7x^2 - 3x \cdot 3x + 3x \cdot 5$	
$21x^3 - 9x^2 + 15x$ ✓	

TRY IT : : 7.19 Factor: $20x^3 - 10x^2 + 14x$.

TRY IT : : 7.20 Factor: $24y^3 - 12y^2 - 20y$.

EXAMPLE 7.11

Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution

Find the GCF of $8m^3$, $12m^2n$, $20mn^2$.	$8m^3 = 2 \cdot 2 \cdot 2 \quad m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot \quad m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot \quad 5 \cdot m \cdot \quad n \cdot n$ GCF $= 2 \cdot 2 \cdot \quad m$ GCF $= 4m$
	$8m^3 - 12m^2n + 20mn^2$
Rewrite each term.	$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Factor the GCF.	$4m(2m^2 - 3mn + 5n^2)$
Check.	
$4m(2m^2 - 3mn + 5n^2)$	
$4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$	
$8m^3 - 12m^2n + 20mn^2$ ✓	

TRY IT : : 7.21 Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

TRY IT : : 7.22 Factor: $3p^3 - 6p^2q + 9pq^3$.

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 7.12

Factor: $-8y - 24$.

Solution

When the leading coefficient is negative, the GCF will be negative.

Ignoring the signs of the terms, we first find the GCF of 8*y* and 24 is 8. Since the expression −8*y* − 24 has a negative leading coefficient, we use −8 as the GCF.	$8y = 2 \cdot 2 \cdot 2 \cdot \quad y$ $24 = 2 \cdot 2 \cdot 2 \cdot 3$ GCF $= 2 \cdot 2 \cdot 2$ GCF $= 8$
Rewrite each term using the GCF.	$-8y - 24$ $-8 \cdot y + (-8) \cdot 3$
Factor the GCF.	$-8(y + 3)$
Check.	
$-8(y + 3)$	
$-8 \cdot y + (-8) \cdot 3$	

$-8y - 24$ ✓

TRY IT : : 7.23 Factor: $-16z - 64$.

TRY IT : : 7.24 Factor: $-9y - 27$.

EXAMPLE 7.13

Factor: $-6a^2 + 36a$.

Solution

The leading coefficient is negative, so the GCF will be negative.?

Since the leading coefficient is negative, the GCF is negative, −6*a*.	$6a^2 = 2 \cdot 3 \cdot a \cdot a$ $36a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a$ $\text{GCF} = 2 \cdot 3 \cdot a$ $\text{GCF} = 6a$ $-6a^2 + 36a$
Rewrite each term using the GCF.	$-6a \cdot a - (-6a) \cdot 6$
Factor the GCF.	$-6a(a - 6)$
Check.	
$-6a(a - 6)$	
$-6a \cdot a + (-6a)(-6)$	
$-6a^2 + 36a$ ✓	

TRY IT : : 7.25 Factor: $-4b^2 + 16b$.

TRY IT : : 7.26 Factor: $-7a^2 + 21a$.

EXAMPLE 7.14

Factor: $5q(q + 7) - 6(q + 7)$.

Solution

The GCF is the binomial $q + 7$.

	$5q(q + 7) - 6(q + 7)$
Factor the GCF, $(q + 7)$.	$(q + 7)(5q - 6)$
Check on your own by multiplying.	

TRY IT : : 7.27 Factor: $4m(m+3)-7(m+3)$.

TRY IT : : 7.28 Factor: $8n(n-4)+5(n-4)$.

Factor by Grouping

When there is no common factor of all the terms of a polynomial, look for a common factor in just some of the terms. When there are four terms, a good way to start is by separating the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts.

(Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.)

EXAMPLE 7.15 HOW TO FACTOR BY GROUPING

Factor: $xy+3y+2x+6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $\underbrace{xy+3y}+\underbrace{2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$y(x+3)+\underbrace{2x+6}$ $y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$. Factor out the common factor.	$y(x+3)+2(x+3)$ $(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT : : 7.29 Factor: $xy+8y+3x+24$.

TRY IT : : 7.30 Factor: $ab+7b+8a+56$.

HOW TO : : FACTOR BY GROUPING.

Step 1. Group terms with common factors.

Step 2. Factor out the common factor in each group.

Step 3. Factor the common factor from the expression.

Step 4. Check by multiplying the factors.

EXAMPLE 7.16

Factor: $x^2+3x-2x-6$.

Solution

There is no GCF in all four terms.	$x^2 + 3x \ -2x - 6$
Separate into two parts.	$\underbrace{x^2 + 3x}\ \underbrace{-2x - 6}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x + 3) - 2(x + 3)$ $(x + 3)(x - 2)$
Check on your own by multiplying.	

TRY IT : : 7.31 Factor: $x^2 + 2x - 5x - 10$.

TRY IT : : 7.32 Factor: $y^2 + 4y - 7y - 28$.

MEDIA : :

Access these online resources for additional instruction and practice with greatest common factors (GFCs) and factoring by grouping.

- **Greatest Common Factor (GCF) (https://openstax.org/l/25GCF1)**
- **Factoring Out the GCF of a Binomial (https://openstax.org/l/25GCF2)**
- **Greatest Common Factor (GCF) of Polynomials (https://openstax.org/l/25GCF3)**

7.1 EXERCISES

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. 8, 18

2. 24, 40

3. 72, 162

4. 150, 275

5. $10a$, 50

6. $5b$, 30

7. $3x,\ 10x^2$

8. $21b^2,\ 14b$

9. $8w^2,\ 24w^3$

10. $30x^2,\ 18x^3$

11. $10p^3q,\ 12pq^2$

12. $8a^2b^3,\ 10ab^2$

13. $12m^2n^3,\ 30m^5n^3$

14. $28x^2y^4,\ 42x^4y^4$

15. $10a^3,\ 12a^2,\ 14a$

16. $20y^3,\ 28y^2,\ 40y$

17. $35x^3,\ 10x^4,\ 5x^5$

18. $27p^2,\ 45p^3,\ 9p^4$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

19. $4x + 20$

20. $8y + 16$

21. $6m + 9$

22. $14p + 35$

23. $9q + 9$

24. $7r + 7$

25. $8m - 8$

26. $4n - 4$

27. $9n - 63$

28. $45b - 18$

29. $3x^2 + 6x - 9$

30. $4y^2 + 8y - 4$

31. $8p^2 + 4p + 2$

32. $10q^2 + 14q + 20$

33. $8y^3 + 16y^2$

34. $12x^3 - 10x$

35. $5x^3 - 15x^2 + 20x$

36. $8m^2 - 40m + 16$

37. $12xy^2 + 18x^2y^2 - 30y^3$

38. $21pq^2 + 35p^2q^2 - 28q^3$

39. $-2x - 4$

40. $-3b + 12$

41. $5x(x + 1) + 3(x + 1)$

42. $2x(x - 1) + 9(x - 1)$

43. $3b(b - 2) - 13(b - 2)$

44. $6m(m - 5) - 7(m - 5)$

Factor by Grouping

In the following exercises, factor by grouping.

45. $xy + 2y + 3x + 6$

46. $mn + 4n + 6m + 24$

47. $uv - 9u + 2v - 18$

48. $pq - 10p + 8q - 80$

49. $b^2 + 5b - 4b - 20$

50. $m^2 + 6m - 12m - 72$

51. $p^2 + 4p - 9p - 36$

52. $x^2 + 5x - 3x - 15$

Mixed Practice

In the following exercises, factor.

53. $-20x-10$

54. $5x^3-x^2+x$

55. $3x^3-7x^2+6x-14$

56. x^3+x^2-x-1

57. $x^2+xy+5x+5y$

58. $5x^3-3x^2-5x-3$

Everyday Math

59. Area of a rectangle The area of a rectangle with length 6 less than the width is given by the expression w^2-6w, where $w=$ width. Factor the greatest common factor from the polynomial.

60. Height of a baseball The height of a baseball t seconds after it is hit is given by the expression $-16t^2+80t+4$. Factor the greatest common factor from the polynomial.

Writing Exercises

61. The greatest common factor of 36 and 60 is 12. Explain what this means.

62. What is the GCF of $y^4, y^5, \text{and } y^{10}$? Write a general rule that tells you how to find the GCF of $y^a, y^b, \text{and} y^c$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential—every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

7.2 Factor Quadratic Trinomials with Leading Coefficient 1

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $x^2 + bxy + cy^2$

Be Prepared!

Before you get started, take this readiness quiz.

1. Multiply: $(x + 4)(x + 5)$.
   If you missed this problem, review Example 6.38.
2. Simplify: ⓐ $-9 + (-6)$ ⓑ $-9 + 6$.
   If you missed this problem, review Example 1.37.
3. Simplify: ⓐ $-9(6)$ ⓑ $-9(-6)$.
   If you missed this problem, review Example 1.46.
4. Simplify: ⓐ $|-5|$ ⓑ $|3|$.
   If you missed this problem, review Example 1.33.

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication—to start with the product and end up with the factors. Let's look at an example of multiplying binomials to refresh your memory.

$$(x + 2)(x + 3) \quad \text{factors}$$
$$F \quad O \quad I \quad L$$
$$x^2 + 3x + 2x + 6$$
$$x^2 + 5x + 6 \quad \text{product}$$

To factor the trinomial means to start with the product, $x^2 + 5x + 6$, and end with the factors, $(x + 2)(x + 3)$. You need to think about where each of the terms in the trinomial came from.

The *first term* came from multiplying the first term in each binomial. So to get x^2 in the product, each binomial must start with an *x*.

$$x^2 + 5x + 6$$
$$(x \quad)(x \quad)$$

The *last term* in the trinomial came from multiplying the last term in each binomial. So the last terms must multiply to 6.

What two numbers multiply to 6?

The factors of 6 could be 1 and 6, or 2 and 3. How do you know which pair to use?

Consider the *middle term*. It came from adding the outer and inner terms.

So the numbers that must have a product of 6 will need a sum of 5. We'll test both possibilities and summarize the results in Table 7.13—the table will be very helpful when you work with numbers that can be factored in many different ways.

Factors of 6	Sum of factors
$1, 6$	$1 + 6 = 7$
$2, 3$	$2 + 3 = 5$

Table 7.13

We see that 2 and 3 are the numbers that multiply to 6 and add to 5. So we have the factors of $x^2 + 5x + 6$. They are $(x + 2)(x + 3)$.

$$\begin{array}{ll} x^2 + 5x + 6 & \text{product} \\ (x + 2)(x + 3) & \text{factors} \end{array}$$

You should check this by multiplying.

Looking back, we started with $x^2 + 5x + 6$, which is of the form $x^2 + bx + c$, where $b = 5$ and $c = 6$. We factored it into two binomials of the form $(x + m)$ and $(x + n)$.

$$\begin{array}{ll} x^2 + 5x + 6 & x^2 + bx + c \\ (x + 2)(x + 3) & (x + m)(x + n) \end{array}$$

To get the correct factors, we found two numbers *m* and *n* whose product is *c* and sum is *b*.

EXAMPLE 7.17 HOW TO FACTOR TRINOMIALS OF THE FORM $x^2 + bx + c$

Factor: $x^2 + 7x + 12$.

Solution

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 7x + 12$ $(x \quad)(x \quad)$
Step 2. Find two numbers *m and n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 12 and add to 7. **Factors of 12** / **Sum of factors** 1, 12 / 1 + 12 = 13 2, 6 / 2 + 6 = 8 3, 4 / 3 + 4 = 7*	
Step 3. Use *m and n* as the last terms of the factors.	Use 3 and 4 as the last terms of the binomials.	$(x + 3)(x + 4)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 4)$ $x^2 + 4x + 3x + 12$ $x^2 + 7x + 12$ ✓

TRY IT : : 7.33 Factor: $x^2 + 6x + 8$.

TRY IT : : 7.34 Factor: $y^2 + 8y + 15$.

Let's summarize the steps we used to find the factors.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $x^2 + bx + c$.

Step 1. Write the factors as two binomials with first terms *x*: $(x \quad)(x \quad)$.

Step 2. Find two numbers *m* and *n* that
Multiply to *c*, $m \cdot n = c$
Add to *b*, $m + n = b$

Step 3. Use *m* and *n* as the last terms of the factors: $(x + m)(x + n)$.

Step 4. Check by multiplying the factors.

EXAMPLE 7.18

Factor: $u^2 + 11u + 24$.

Solution

Notice that the variable is *u*, so the factors will have first terms *u*.

	$u^2 + 11u + 24$
Write the factors as two binomials with fir t terms u.	$(u \quad)(u \quad)$

Find two numbers that: multiply to 24 and add to 11.

Factors of 24	Sum of factors
$1, 24$	$1 + 24 = 25$
$2, 12$	$2 + 12 = 14$
$3, 8$	$3 + 8 = 11^*$
$4, 6$	$4 + 6 = 10$

Use 3 and 8 as the last terms of the binomials.	$(u + 3)(u + 8)$

Check.

$$(u + 3)(u + 8)$$
$$u^2 + 3u + 8u + 24$$
$$u^2 + 11u + 24 \checkmark$$

TRY IT : : 7.35 Factor: $q^2 + 10q + 24$.

TRY IT : : 7.36 Factor: $t^2 + 14t + 24$.

EXAMPLE 7.19

Factor: $y^2 + 17y + 60$.

Solution

	$y^2 + 17y + 60$
Write the factors as two binomials with fir t terms y.	$(y \quad)(y \quad)$

Find two numbers that multiply to 60 and add to 17.

Factors of 60	Sum of factors
$1, 60$	$1+60=61$
$2, 30$	$2+30=32$
$3, 20$	$3+20=23$
$4, 15$	$4+15=19$
$5, 12$	$5+12=17^*$
$6, 10$	$6+10=16$

Use 5 and 12 as the last terms. $(y+5)(y+12)$

Check.

$$(y+5)(y+12)$$
$$(y^2+12y+5y+60)$$
$$(y^2+17y+60)\ ✓$$

TRY IT : : 7.37 Factor: $x^2+19x+60$.

TRY IT : : 7.38 Factor: $v^2+23v+60$.

Factor Trinomials of the Form $x^2 + bx + c$ with b Negative, c Positive

In the examples so far, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

Remember: To get a negative sum and a positive product, the numbers must both be negative.

Again, think about FOIL and where each term in the trinomial came from. Just as before,

- the first term, x^2, comes from the product of the two first terms in each binomial factor, *x* and *y*;
- the positive last term is the product of the two last terms
- the negative middle term is the sum of the outer and inner terms.

How do you get a *positive product* and a *negative sum*? With two negative numbers.

EXAMPLE 7.20

Factor: $t^2-11t+28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11t$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

$$t^2-11t+28$$

Write the factors as two binomials with fir t terms t. $(t\quad)(t\quad)$

Find two numbers that: multiply to 28 and add to -11.

Factors of 28	Sum of factors
$-1, -28$	$-1+(-28)=-29$
$-2, -14$	$-2+(-14)=-16$
$-4, -7$	$-4+(-7)=-11^*$

Use $-4,\ -7$ as the last terms of the binomials. $(t-4)(t-7)$

Check.

$(t-4)(t-7)$

$t^2-7t-4t+28$

$t^2-11t+28$ ✓

> **TRY IT : :** 7.39 Factor: $u^2-9u+18$.

> **TRY IT : :** 7.40 Factor: $y^2-16y+63$.

Factor Trinomials of the Form x^2+bx+c with c Negative

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

Remember: To get a negative product, the numbers must have different signs.

EXAMPLE 7.21

Factor: z^2+4z-5.

Solution

To get a negative last term, multiply one positive and one negative. We need factors of -5 that add to positive 4.

Factors of -5	Sum of factors
$1, -5$	$1+(-5)=-4$
$-1, 5$	$-1+5=4^*$

Notice: We listed both $1, -5$ and $-1, 5$ to make sure we got the sign of the middle term correct.

	z^2+4z-5
Factors will be two binomials with fir t terms z.	$(z\quad)(z\quad)$
Use $-1,\ 5$ as the last terms of the binomials.	$(z-1)(z+5)$

Check.

$(z-1)(z+5)$

$z^2+5z-1z-5$

z^2+4z-5 ✓

TRY IT : : 7.41 Factor: $h^2 + 4h - 12$.

TRY IT : : 7.42 Factor: $k^2 + k - 20$.

Let's make a minor change to the last trinomial and see what effect it has on the factors.

EXAMPLE 7.22

Factor: $z^2 - 4z - 5$.

Solution

This time, we need factors of -5 that add to -4.

Factors of -5	Sum of factors
$1, -5$	$1 + (-5) = -4$*
$-1, 5$	$-1 + 5 = 4$

	$z^2 - 4z - 5$
Factors will be two binomials with fir t terms z.	$(z \quad)(z \quad)$
Use $1, -5$ as the last terms of the binomials.	$(z + 1)(z - 5)$
Check.	

$(z + 1)(z - 5)$
$z^2 - 5z + 1z - 5$
$z^2 - 4z - 5$ ✓

Notice that the factors of $z^2 - 4z - 5$ are very similar to the factors of $z^2 + 4z - 5$. It is very important to make sure you choose the factor pair that results in the correct sign of the middle term.

TRY IT : : 7.43 Factor: $x^2 - 4x - 12$.

TRY IT : : 7.44 Factor: $y^2 - y - 20$.

EXAMPLE 7.23

Factor: $q^2 - 2q - 15$.

Solution

	$q^2 - 2q - 15$
Factors will be two binomials with fir t terms q.	$(q \quad)(q \quad)$
You can use $3, -5$ as the last terms of the binomials.	$(q + 3)(q - 5)$

Factors of -15	Sum of factors
$1, -15$	$1+(-15)=-14$
$-1, 15$	$-1+15=14$
$3, -5$	$3+(-5)=-2$*
$-3, 5$	$-3+5=2$

Check.

$(q+3)(q-5)$

$q^2-5q+3q-15$

$q^2-2q-15$ ✓

TRY IT : : 7.45 Factor: $r^2-3r-40$.

TRY IT : : 7.46 Factor: $s^2-3s-10$.

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 7.24

Factor: $y^2-6y+15$.

Solution

	$y^2-6y+15$
Factors will be two binomials with fir t terms y.	$(y\quad)(y\quad)$

Factors of 15	Sum of factors
$-1, -15$	$-1+(-15)=-16$
$-3, -5$	$-3+(-5)=-8$

As shown in the table, none of the factors add to -6; therefore, the expression is prime.

TRY IT : : 7.47 Factor: $m^2+4m+18$.

TRY IT : : 7.48 Factor: $n^2-10n+12$.

EXAMPLE 7.25

Factor: $2x+x^2-48$.

Solution

	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with fir t terms x.	$(x \quad)(x \quad)$

As shown in the table, you can use $-6,\ 8$ as the last terms of the binomials.

$$(x - 6)(x + 8)$$

Factors of -48	Sum of factors
$-1,\ 48$	$-1 + 48 = 47$
$-2,\ 24$ $-3,\ 16$ $-4,\ 12$ $-6,\ 8$	$-2 + 24 = 22$ $-3 + 16 = 13$ $-4 + 12 = 8$ $-6 + 8 = 2$

Check.

$$(x - 6)(x + 8)$$

$$x^2 - 6q + 8q - 48$$

$$x^2 + 2x - 48\ ✓$$

TRY IT : : 7.49 Factor: $9m + m^2 + 18$.

TRY IT : : 7.50 Factor: $-7n + 12 + n^2$.

Let’s summarize the method we just developed to factor trinomials of the form $x^2 + bx + c$.

HOW TO : : FACTOR TRINOMIALS.

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When *c* is positive, *m* and *n* have the same sign.

b positive	b negative
$m,\ n$ positive	$m,\ n$ negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When *c* is negative, *m* and *n* have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

Factor Trinomials of the Form $x^2 + bxy + cy^2$

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in the previous objective.

EXAMPLE 7.26

Factor: $x^2 + 12xy + 36y^2$.

Solution

	$x^2 + 12xy + 36y^2$
Note that the fir t terms are x, last terms contain y.	$(x\_y)(x\_y)$

Find the numbers that multiply to 36 and add to 12.

Factors of 36	Sum of factors
1, 36	$1 + 36 = 37$
2, 18	$2 + 18 = 20$
3, 12	$3 + 12 = 15$
4, 9	$4 + 9 = 13$
6, 6	$6 + 6 = 12^*$

Use 6 and 6 as the coefficients of he last terms. $(x+6y)(x+6y)$

Check your answer.

$$(x+6y)(x+6y)$$
$$x^2+6xy+6xy+36y^2$$
$$x^2+12xy+36y^2 \checkmark$$

TRY IT : : 7.51 Factor: $u^2+11uv+28v^2$.

TRY IT : : 7.52 Factor: $x^2+13xy+42y^2$.

EXAMPLE 7.27

Factor: $r^2-8rx-9s^2$.

Solution

We need r in the first term of each binomial and s in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$r^2-8rx-9s^2$
Note that the fir t terms are r, last terms contain s.	$(r\_s)(r\_s)$

Find the numbers that multiply to -9 and add to -8.

Factors of -9	Sum of factors
$1, -9$	$1+(-9)=-8$*
$-1, 9$	$-1+9=8$
$3, -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of he last terms. $(r+s)(r-9s)$

Check your answer.

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

TRY IT : : 7.53 Factor: $a^2-11ab+10b^2$.

TRY IT : : 7.54 Factor: $m^2-13mn+12n^2$.

EXAMPLE 7.28

Factor: $u^2-9uv-12v^2$.

Solution

We need u in the first term of each binomial and v in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$u^2 - 9uv - 12v^2$
Note that the fir t terms are u, last terms contain v.	$(u\_v)(u\_v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1 + (-12) = -11$
$-1, 12$	$-1 + 12 = 11$
$2, -6$	$2 + (-6) = -4$
$-2, 6$	$-2 + 6 = 4$
$3, -4$	$3 + (-4) = -1$
$-3, 4$	$-3 + 4 = 1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT : : 7.55 Factor: $x^2 - 7xy - 10y^2$.

TRY IT : : 7.56 Factor: $p^2 + 15pq + 20q^2$.

7.2 EXERCISES

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

63. $x^2 + 4x + 3$

64. $y^2 + 8y + 7$

65. $m^2 + 12m + 11$

66. $b^2 + 14b + 13$

67. $a^2 + 9a + 20$

68. $m^2 + 7m + 12$

69. $p^2 + 11p + 30$

70. $w^2 + 10x + 21$

71. $n^2 + 19n + 48$

72. $b^2 + 14b + 48$

73. $a^2 + 25a + 100$

74. $u^2 + 101u + 100$

75. $x^2 - 8x + 12$

76. $q^2 - 13q + 36$

77. $y^2 - 18x + 45$

78. $m^2 - 13m + 30$

79. $x^2 - 8x + 7$

80. $y^2 - 5y + 6$

81. $p^2 + 5p - 6$

82. $n^2 + 6n - 7$

83. $y^2 - 6y - 7$

84. $v^2 - 2v - 3$

85. $x^2 - x - 12$

86. $r^2 - 2r - 8$

87. $a^2 - 3a - 28$

88. $b^2 - 13b - 30$

89. $w^2 - 5w - 36$

90. $t^2 - 3t - 54$

91. $x^2 + x + 5$

92. $x^2 - 3x - 9$

93. $8 - 6x + x^2$

94. $7x + x^2 + 6$

95. $x^2 - 12 - 11x$

96. $-11 - 10x + x^2$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

97. $p^2 + 3pq + 2q^2$

98. $m^2 + 6mn + 5n^2$

99. $r^2 + 15rs + 36s^2$

100. $u^2 + 10uv + 24v^2$

101. $m^2 - 12mn + 20n^2$

102. $p^2 - 16pq + 63q^2$

103. $x^2 - 2xy - 80y^2$

104. $p^2 - 8pq - 65q^2$

105. $m^2 - 64mn - 65n^2$

106. $p^2 - 2pq - 35q^2$

107. $a^2 + 5ab - 24b^2$

108. $r^2 + 3rs - 28s^2$

109. $x^2 - 3xy - 14y^2$

110. $u^2 - 8uv - 24v^2$

111. $m^2 - 5mn + 30n^2$

112. $c^2 - 7cd + 18d^2$

Mixed Practice

In the following exercises, factor each expression.

113. $u^2 - 12u + 36$

114. $w^2 + 4w - 32$

115. $x^2 - 14x - 32$

116. $y^2 + 41y + 40$

117. $r^2 - 20rs + 64s^2$

118. $x^2 - 16xy + 64y^2$

119. $k^2 + 34k + 120$

120. $m^2 + 29m + 120$

121. $y^2 + 10y + 15$

122. $z^2 - 3z + 28$

123. $m^2 + mn - 56n^2$

124. $q^2 - 29qr - 96r^2$

125. $u^2 - 17uv + 30v^2$

126. $m^2 - 31mn + 30n^2$

127. $c^2 - 8cd + 26d^2$

128. $r^2 + 11rs + 36s^2$

Everyday Math

129. Consecutive integers Deirdre is thinking of two consecutive integers whose product is 56. The trinomial $x^2 + x - 56$ describes how these numbers are related. Factor the trinomial.

130. Consecutive integers Deshawn is thinking of two consecutive integers whose product is 182. The trinomial $x^2 + x - 182$ describes how these numbers are related. Factor the trinomial.

Writing Exercises

131. Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

132. How do you determine whether to use plus or minus signs in the binomial factors of a trinomial of the form $x^2 + bx + c$ where b and c may be positive or negative numbers?

133. Will factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Bill factored it as $(x + 4)(x - 5)$. Phil factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

134. Look at Example 7.19, where we factored $y^2 + 17y + 60$. We made a table listing all pairs of factors of 60 and their sums. Do you find this kind of table helpful? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$.			
factor trinomials of the form $x^2 + bxy + cy^2$.			

ⓑ *After reviewing this checklist, what will you do to become confident for all goals?*

7.3 Factor Quadratic Trinomials with Leading Coefficient Other than 1

Learning Objectives

By the end of this section, you will be able to:

- Recognize a preliminary strategy to factor polynomials completely
- Factor trinomials of the form $ax^2 + bx + c$ with a GCF
- Factor trinomials using trial and error
- Factor trinomials using the ‘ac’ method

Be Prepared!

Before you get started, take this readiness quiz.

1. Find the GCF of $45p^2$ and $30p^6$.
   If you missed this problem, review **Example 7.2**.
2. Multiply $(3y + 4)(2y + 5)$.
   If you missed this problem, review **Example 6.40**.
3. Combine like terms $12x^2 + 3x + 5x + 9$.
   If you missed this problem, review **Example 1.24**.

Recognize a Preliminary Strategy for Factoring

Let’s summarize where we are so far with factoring polynomials. In the first two sections of this chapter, we used three methods of factoring: factoring the GCF, factoring by grouping, and factoring a trinomial by “undoing” FOIL. More methods will follow as you continue in this chapter, as well as later in your studies of algebra.

How will you know when to use each factoring method? As you learn more methods of factoring, how will you know when to apply each method and not get them confused? It will help to organize the factoring methods into a strategy that can guide you to use the correct method.

As you start to factor a polynomial, always ask first, “Is there a greatest common factor?” If there is, factor it first.

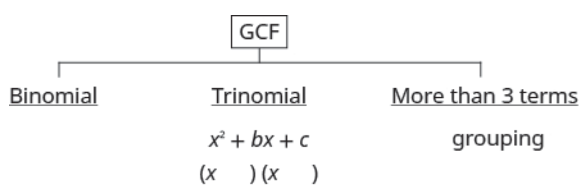
The next thing to consider is the type of polynomial. How many terms does it have? Is it a binomial? A trinomial? Or does it have more than three terms?

If it is a trinomial where the leading coefficient is one, $x^2 + bx + c$, use the “undo FOIL” method.

If it has more than three terms, try the grouping method. This is the only method to use for polynomials of more than three terms.

Some polynomials cannot be factored. They are called “prime.”

Below we summarize the methods we have so far. These are detailed in **Choose a strategy to factor polynomials completely**.



HOW TO : : CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY.

Step 1. Is there a greatest common factor?
- Factor it out.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial, right now we have no method to factor it.
- If it is a trinomial of the form $x^2 + bx + c$: Undo FOIL $(x \quad)(x \quad)$
- If it has more than three terms: Use the grouping method.

Step 3. Check by multiplying the factors.

Use the preliminary strategy to completely factor a polynomial. A polynomial is factored completely if, other than monomials, all of its factors are prime.

EXAMPLE 7.29

Identify the best method to use to factor each polynomial.

ⓐ $6y^2 - 72$ ⓑ $r^2 - 10r - 24$ ⓒ $p^2 + 5p + pq + 5q$

Solution

ⓐ

	$6y^2 - 72$
Is there a greatest common factor?	Yes, 6.
Factor out the 6.	$6\left(y^2 - 12\right)$
Is it a binomial, trinomial, or are there more than 3 terms?	Binomial, we have no method to factor binomials yet.

ⓑ

	$r^2 - 10r - 24$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial, with leading coefficient 1, "undo" FOIL.

ⓒ

	$p^2 + 5p + pq + 5q$
Is there a greatest common factor?	No, there is no common factor.
Is it a binomial, trinomial, or are there more than three terms?	More than three terms, so factor using grouping.

TRY IT : : 7.57 Identify the best method to use to factor each polynomial:

ⓐ $4y^2 + 32$ ⓑ $y^2 + 10y + 21$ ⓒ $yz + 2y + 3z + 6$

TRY IT : : 7.58 Identify the best method to use to factor each polynomial:

ⓐ $ab + a + 4b + 4$ ⓑ $3k^2 + 15$ ⓒ $p^2 + 9p + 8$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

Now that we have organized what we've covered so far, we are ready to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods in the last section. Let's do a few examples to see how this works.

Watch out for the signs in the next two examples.

EXAMPLE 7.30

Factor completely: $2n^2 - 8n - 42$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$2n^2 - 8n - 42$
Yes, GCF = 2. Factor it out.	$2\left(n^2 - 4n - 21\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms?	
It is a trinomial whose coefficient is 1, so undo FOIL.	$2(n\quad)(n\quad)$
Use 3 and −7 as the last terms of the binomials.	$2(n + 3)(n - 7)$

Factors of -21	Sum of factors
$1, -21$	$1 + (-21) = -20$
$3, -7$	$3 + (-7) = -4$*

Check.

$2(n + 3)(n - 7)$

$2(n^2 - 7n + 3n - 21)$

$2(n^2 - 4n - 21)$

$2n^2 - 8n - 42$ ✓

TRY IT : : 7.59 Factor completely: $4m^2 - 4m - 8$.

TRY IT : : 7.60 Factor completely: $5k^2 - 15k - 50$.

EXAMPLE 7.31

Factor completely: $4y^2 - 36y + 56$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4y^2 - 36y + 56$
Yes, GCF = 4. Factor it.	$4\left(y^2 - 9y + 14\right)$
Inside the parentheses, is it a binomial, trinomial, or are there more than three terms? It is a trinomial whose coefficient is 1. So undo OIL.	$4(y \quad)(y \quad)$
Use a table like the one below to find t o numbers that multiply to 14 and add to −9. Both factors of 14 must be negative.	$4(y - 2)(y - 7)$

Factors of 14	Sum of factors
$-1, -14$	$-1 + (-14) = -15$
$-2, -7$	$-2 + (-7) = -9^*$

Check.

$4(y - 2)(y - 7)$

$4(y^2 - 7y - 2y + 14)$

$4(y^2 - 9y + 14)$

$4y^2 - 36y + 42$ ✓

TRY IT : : 7.61 Factor completely: $3r^2 - 9r + 6$.

TRY IT : : 7.62 Factor completely: $2t^2 - 10t + 12$.

In the next example the GCF will include a variable.

EXAMPLE 7.32

Factor completely: $4u^3 + 16u^2 - 20u$.

Solution

Use the preliminary strategy.

Is there a greatest common factor?	$4u^3 + 16u^2 - 20u$
Yes, GCF = $4u$. Factor it.	$4u(u^2 + 4u - 5)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4u(u \quad)(u \quad)$
Use a table like the table below to find t o numbers that multiply to -5 and add to 4.	$4u(u - 1)(u + 5)$

Factors of -5	Sum of factors
$-1, 5$	$-1 + 5 = 4$*
$1, -5$	$1 + (-5) = -4$

Check.

$$4u(u - 1)(u + 5)$$

$$4u(u^2 + 5u - u - 5)$$

$$4u(u^2 + 4u - 5)$$

$$4u^3 + 16u^2 - 20u \checkmark$$

TRY IT : : 7.63 Factor completely: $5x^3 + 15x^2 - 20x$.

TRY IT : : 7.64 Factor completely: $6y^3 + 18y^2 - 60y$.

Factor Trinomials using Trial and Error

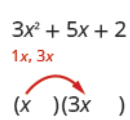
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work we expect this will factor into two binomials.

$$3x^2 + 5x + 2$$
$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.



Check. Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2, or 2, 1.

$3x^2 + 5x + 2$ (1x, 3x; 1, 2) → $(x + 1)(3x + 2)$ or $3x^2 + 5x + 2$ (1x, 3x; 1, 2) → $(x + 2)(3x + 1)$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 1)(3x + 2)$: $3x$, $2x$, total $5x$ or $3x^2 + 5x + 2$ (1x, 3x; 1, 2): $(x + 2)(3x + 1)$: $6x$, $1x$, total $7x$

Since the middle term of the trinomial is 5*x*, the factors in the first case will work. Let's FOIL to check.

$$(x + 1)(3x + 2)$$
$$3x^2 + 2x + 3x + 2$$
$$3x^2 + 5x + 2\ ✓$$

Our result of the factoring is:

$$3x^2 + 5x + 2$$
$$(x + 1)(3x + 2)$$

EXAMPLE 7.33 HOW TO FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR

Factor completely: $3y^2 + 22y + 7$.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Find all the factor pairs of the first term.	The only factors of $3y^2$ are 1*y*, 3*y* Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ (1y, 3y) $3y^2 + 22y + 7$ (1y, 3y) $(y\quad)(3y\quad)$
Step 3. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ (1y, 3y; 1, 7) $(y\quad)(3y\quad)$
Step 4. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ (1y, 3y; 1, 7) $(y + 1)(3y + 7)$: $3y$, $7y$, total $10y$ No. We need 22*y* $3y^2 + 22y + 7$ (1y, 3y; 1, 7) $(y + 7)(3y + 1)$: $21y$, y, total $22y$	$3y^2 + 22y + 7$: Possible factors / Product $(y + 1)(3y + 7)$ / $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ / $3y^2 + 22y + 7$ $(y + 7)(3y + 1)$

Step 5. Check by multiplying.		$(y+7)(3y+1)$ $3y^2+22y+7$ ✓

> TRY IT : : 7.65 Factor completely: $2a^2+5a+3$.

> TRY IT : : 7.66 Factor completely: $4b^2+5b+1$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM ax^2+bx+c USING TRIAL AND ERROR.

Step 1. Write the trinomial in descending order of degrees.
Step 2. Find all the factor pairs of the first term.
Step 3. Find all the factor pairs of the third term.
Step 4. Test all the possible combinations of the factors until the correct product is found.
Step 5. Check by multiplying.

When the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

EXAMPLE 7.34

Factor completely: $6b^2-13b+5$.

Solution

The trinomial is already in descending order.	$6b^2-13b+5$
Find the factors of the first term.	$6b^2-13b+5$ $1b\cdot 6b$ $2b\cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5 is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2-13b+5$ $1b\cdot 6b$ $-1, -5$ $2b\cdot 3b$

Consider all the combinations of factors.

$6b^2-13b+5$	
Possible factors	**Product**
$(b-1)(6b-5)$	$6b^2-11b+5$
$(b-5)(6b-1)$	$6b^2-31b+5$
$(2b-1)(3b-5)$	$6b^2-13b+5$ *
$(2b-5)(3b-1)$	$6b^2-17b+5$

The correct factors are those whose product is the original trinomial.	$(2b-1)(3b-5)$

Check by multiplying.

$$(2b-1)(3b-5)$$
$$6b^2-10b-3b+5$$
$$6b^2-13b+5\ ✓$$

TRY IT : : 7.67 Factor completely: $8x^2-13x+3$.

TRY IT : : 7.68 Factor completely: $10y^2-37y+7$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 7.35

Factor completely: $14x^2-47x-7$.

Solution

The trinomial is already in descending order.	$14x^2-47x-7$
Find the factors of the first term.	$14x^2-47x-7$ $1x \cdot 14x$ $2x \cdot 7x$
Find the factors of the last term. Consider the signs. Since it is negative, one factor must be positive and one negative.	$14x^2-47x-7$ $1x \cdot 14x$ $1,-7$ $2x \cdot 7x$ $-1,7$

Consider all the combinations of factors. We use each pair of the factors of $14x^2$ with each pair of factors of -7.

Factors of $14x^2$	Pair with	Factors of -7
$x,\ 14x$		$1,\ -7$ $-7,\ 1$ (reverse order)
$x,\ 14x$		$-1,\ 7$ $7,\ -1$ (reverse order)
$2x,\ 7x$		$1,\ -7$ $-7,\ 1$ (reverse order)
$2x,\ 7x$		$-1,\ 7$ $7,\ -1$ (reverse order)

These pairings lead to the following eight combinations.

$14x^2-47x-7$	
Possible factors	**Product**
$(x+1)(14x-7)$	Not an option
$(x-7)(14x+1)$	$14x^2-97x-7$
$(x-1)(14x+7)$	Not an option
$(x+7)(14x-1)$	$14x^2+97x-7$
$(2x+1)(7x-7)$	Not an option
$(2x-7)(7x+1)$	$14x^2-47x-7$*
$(2x-1)(7x+7)$	Not an option
$(2x+7)(7x-1)$	$14x^2+47x-7$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

The correct factors are those whose product is the original trinomial. $(2x-7)(7x+1)$

Check by multiplying.

$$(2x-7)(7x+1)$$
$$14x^2+2x-49x-7$$
$$14x^2-47x-7\ ✓$$

TRY IT : : 7.69 Factor completely: $8a^2-3a-5$.

TRY IT : : 7.70 Factor completely: $6b^2-b-15$.

EXAMPLE 7.36

Factor completely: $18n^2-37n+15$.

Solution

The trinomial is already in descending order.	$18n^2-37n+15$
Find the factors of the first term.	$18n^2-37n+15$ $1n\cdot18n$ $2n\cdot9n$ $3n\cdot6n$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative facotrs.	$18n^2-37n+15$ $1n\cdot18n$ $-1(-15)$ $2n\cdot9n$ $-3(-5)$ $3n\cdot6n$

Consider all the combinations of factors.

$18n^2-37n+15$	
Possible factors	**Product**
$(n-1)(18n-15)$	Not an option
$(n-15)(18n-1)$	$18n^2-271n+15$
$(n-3)(18n-5)$	$18n^2-59n+15$
$(n-5)(18n-3)$	Not an option
$(2n-1)(9n-15)$	Not an option
$(2n-15)(9n-1)$	$18n^2-137n+15$
$(2n-3)(9n-5)$	$18n^2-37n+15$*
$(2n-5)(9n-3)$	Not an option
$(3n-1)(6n-15)$	Not an option
$(3n-15)(6n-1)$	Not an option
$(3n-3)(6n-5)$	Not an option
$(3n-5)(6n-3)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial.	$(2n-3)(9n-5)$

Check by multiplying.

$$(2n-3)(9n-5)$$
$$18n^2-10n-27n+15$$
$$18n^2-37n+15\ ✓$$

TRY IT : : 7.71 Factor completely: $18x^2-3x-10$.

TRY IT : : 7.72 Factor completely: $30y^2-53y-21$.

Don't forget to look for a GCF first.

EXAMPLE 7.37

Factor completely: $10y^4+55y^3+60y^2$.

Solution

	$10y^4+55y^3+60y^2$
Notice the greatest common factor, and factor it first.	$15y^2(2y^2+11y+12)$
Factor the trinomial.	$5y^2(2y^2+11y+12)$ $y\cdot 2y$ $1\cdot 12$, $2\cdot 6$, $3\cdot 4$

Consider all the combinations.

$2y^2+11y+12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $5y^2$.	$5y^2(y+4)(2y+3)$

Check by multiplying.

$$5y^2(y+4)(2y+3)$$
$$5y^2(2y^2+8y+3y+12)$$
$$10y^4+55y^3+60y^2\ ✓$$

TRY IT : : 7.73 Factor completely: $15n^3-85n^2+100n$.

TRY IT : : 7.74 Factor completely: $56q^3 + 320q^2 - 96q$.

Factor Trinomials using the “ac” Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the “ac” method. (The “ac” method is sometimes called the grouping method.) The “ac” method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

EXAMPLE 7.38 HOW TO FACTOR TRINOMIALS USING THE “AC” METHOD

Factor: $6x^2 + 7x + 2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No.	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers *m* and *n* that: Multiply to *ac* $m \cdot n = a \cdot c$ Add to *b* $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12$ $3 + 4 = 7$	
Step 4. Split the middle term using *m*, and *n* $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

TRY IT : : 7.75 Factor: $6x^2 + 13x + 2$.

TRY IT : : 7.76 Factor: $4y^2 + 8y + 3$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM USING THE "AC" METHOD.

Step 1. Factor any GCF.

Step 2. Find the product ac.

Step 3. Find two numbers *m* and *n* that:

Multiply to ac $\quad m \cdot n = a \cdot c$

Add to b $\quad m + n = b$

Step 4. Split the middle term using *m* and *n*:

$$ax^2 + bx + c$$
$$ax^2 + \underbrace{mx + nx}_{bx} + c$$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

When the third term of the trinomial is negative, the factors of the third term will have opposite signs.

EXAMPLE 7.39

Factor: $8u^2 - 17u - 21$.

Solution

Is there a greatest common factor? No.	$ax^2 + bx + c$ $8u^2 - 17u - 21$
Find $a \cdot c$.	$a \cdot c$
	$8(-21)$
	-168

Find two numbers that multiply to -168 and add to -17 . The larger factor must be negative.

Factors of -168	Sum of factors
$1, -168$	$1 + (-168) = -167$
$2, -84$	$2 + (-84) = -82$
$3, -56$	$3 + (-56) = -53$
$4, -42$	$4 + (-42) = -38$
$6, -28$	$6 + (-28) = -22$
$7, -24$	$7 + (-24) = -17$*
$8, -21$	$8 + (-21) = -13$

Split the middle term using $7u$ and $-24u$.	$8u^2 - 17u - 21$ $\underbrace{8u^2 + 7u}\underbrace{-24u - 21}$
Factor by grouping.	$u(8u + 7) - 3(8u + 7)$ $(8u + 7)(u - 3)$

Check by multiplying.

$(8u + 7)(u - 3)$
$8u^2 - 24u + 7u - 21$
$8u^2 - 17u - 21$ ✓

TRY IT : : 7.77 Factor: $20h^2 + 13h - 15$.

TRY IT : : 7.78 Factor: $6g^2 + 19g - 20$.

EXAMPLE 7.40

Factor: $2x^2 + 6x + 5$.

Solution

Is there a greatest common factor? No.	$\overset{ax^2 + bx + c}{2x^2 + 6x + 5}$
Find $a \cdot c$.	$a \cdot c$
	$2(5)$
	10

Find two numbers that multiply to 10 and add to 6.

Factors of 10	Sum of factors
$1,\ 10$	$1 + 10 = 11$
2, 5	$2 + 5 = 7$

There are no factors that multiply to 10 and add to 6. The polynomial is prime.

TRY IT : : 7.79 Factor: $10t^2 + 19t - 15$.

TRY IT : : 7.80 Factor: $3u^2 + 8u + 5$.

Don't forget to look for a common factor!

EXAMPLE 7.41

Factor: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor? Yes. The GCF is 5.	$10y^2 - 55y + 70$
Factor it. Be careful to keep the factor of 5 all the way through the solution!	$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.	$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Factor the trinomial.	$5(y - 2)(2y - 7)$
Check by mulitplying all three factors.	
$5\left(2y^2 - 2y - 4y + 14\right)$	
$5\left(2y^2 - 11y + 14\right)$	
$10y^2 - 55y + 70$ ✓	

TRY IT : : 7.81 Factor: $16x^2 - 32x + 12$.

TRY IT : : 7.82 Factor: $18w^2 - 39w + 18$.

We can now update the Preliminary Factoring Strategy, as shown in **Figure 7.2** and detailed in **Choose a strategy to factor polynomials completely (updated)**, to include trinomials of the form $ax^2 + bx + c$. Remember, some polynomials are prime and so they cannot be factored.

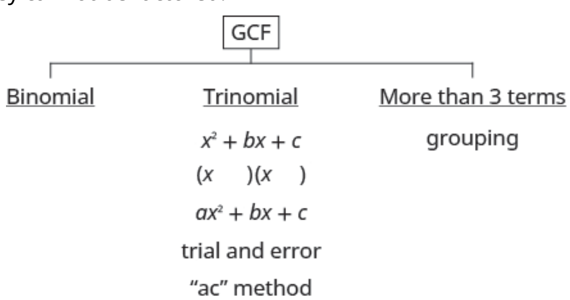


Figure 7.2

HOW TO : : CHOOSE A STRATEGY TO FACTOR POLYNOMIALS COMPLETELY (UPDATED).

Step 1. Is there a greatest common factor?
  - Factor it.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
  - If it is a binomial, right now we have no method to factor it.
  - If it is a trinomial of the form $x^2 + bx + c$
    Undo FOIL $(x \quad)(x \quad)$.
  - If it is a trinomial of the form $ax^2 + bx + c$
    Use Trial and Error or the "ac" method.
  - If it has more than three terms
    Use the grouping method.

Step 3. Check by multiplying the factors.

MEDIA : :

Access these online resources for additional instruction and practice with factoring trinomials of the form $ax^2 + bx + c$.

- **Factoring Trinomials, a is not 1 (https://openstax.org/l/25FactorTrinom)**

7.3 EXERCISES

Practice Makes Perfect

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

135.
ⓐ $10q^2 + 50$
ⓑ $a^2 - 5a - 14$
ⓒ $uv + 2u + 3v + 6$

136.
ⓐ $n^2 + 10n + 24$
ⓑ $8u^2 + 16$
ⓒ $pq + 5p + 2q + 10$

137.
ⓐ $x^2 + 4x - 21$
ⓑ $ab + 10b + 4a + 40$
ⓒ $6c^2 + 24$

138.
ⓐ $20x^2 + 100$
ⓑ $uv + 6u + 4v + 24$
ⓒ $y^2 - 8y + 15$

Factor Trinomials of the form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

139. $5x^2 + 35x + 30$

140. $12s^2 + 24s + 12$

141. $2z^2 - 2z - 24$

142. $3u^2 - 12u - 36$

143. $7v^2 - 63v + 56$

144. $5w^2 - 30w + 45$

145. $p^3 - 8p^2 - 20p$

146. $q^3 - 5q^2 - 24q$

147. $3m^3 - 21m^2 + 30m$

148. $11n^3 - 55n^2 + 44n$

149. $5x^4 + 10x^3 - 75x^2$

150. $6y^4 + 12y^3 - 48y^2$

Factor Trinomials Using Trial and Error

In the following exercises, factor.

151. $2t^2 + 7t + 5$

152. $5y^2 + 16y + 11$

153. $11x^2 + 34x + 3$

154. $7b^2 + 50b + 7$

155. $4w^2 - 5w + 1$

156. $5x^2 - 17x + 6$

157. $6p^2 - 19p + 10$

158. $21m^2 - 29m + 10$

159. $4q^2 - 7q - 2$

160. $10y^2 - 53y - 11$

161. $4p^2 + 17p - 15$

162. $6u^2 + 5u - 14$

163. $16x^2 - 32x + 16$

164. $81a^2 + 153a - 18$

165. $30q^3 + 140q^2 + 80q$

166. $5y^3 + 30y^2 - 35y$

Factor Trinomials using the ‘ac’ Method

In the following exercises, factor.

167. $5n^2 + 21n + 4$

168. $8w^2 + 25w + 3$

169. $9z^2 + 15z + 4$

170. $3m^2 + 26m + 48$

171. $4k^2 - 16k + 15$

172. $4q^2 - 9q + 5$

173. $5s^2 - 9s + 4$

174. $4r^2 - 20r + 25$

175. $6y^2 + y - 15$

176. $6p^2 + p - 22$

177. $2n^2 - 27n - 45$

178. $12z^2 - 41z - 11$

179. $3x^2 + 5x + 4$

180. $4y^2 + 15y + 6$

181. $60y^2 + 290y - 50$

182. $6u^2 - 46u - 16$

183. $48z^3 - 102z^2 - 45z$

184. $90n^3 + 42n^2 - 216n$

185. $16s^2 + 40s + 24$

186. $24p^2 + 160p + 96$

187. $48y^2 + 12y - 36$

188. $30x^2 + 105x - 60$

Mixed Practice

In the following exercises, factor.

189. $12y^2 - 29y + 14$

190. $12x^2 + 36y - 24z$

191. $a^2 - a - 20$

192. $m^2 - m - 12$

193. $6n^2 + 5n - 4$

194. $12y^2 - 37y + 21$

195. $2p^2 + 4p + 3$

196. $3q^2 + 6q + 2$

197. $13z^2 + 39z - 26$

198. $5r^2 + 25r + 30$

199. $x^2 + 3x - 28$

200. $6u^2 + 7u - 5$

201. $3p^2 + 21p$

202. $7x^2 - 21x$

203. $6r^2 + 30r + 36$

204. $18m^2 + 15m + 3$

205. $24n^2 + 20n + 4$

206. $4a^2 + 5a + 2$

207. $x^2 + 2x - 24$

208. $2b^2 - 7b + 4$

Everyday Math

209. Height of a toy rocket The height of a toy rocket launched with an initial speed of 80 feet per second from the balcony of an apartment building is related to the number of seconds, *t*, since it is launched by the trinomial $-16t^2 + 80t + 96$. Completely factor the trinomial.

210. Height of a beach ball The height of a beach ball tossed up with an initial speed of 12 feet per second from a height of 4 feet is related to the number of seconds, *t*, since it is tossed by the trinomial $-16t^2 + 12t + 4$. Completely factor the trinomial.

Writing Exercises

211. List, in order, all the steps you take when using the “ac” method to factor a trinomial of the form $ax^2 + bx + c$.

212. How is the “ac” method similar to the “undo FOIL” method? How is it different?

213. What are the questions, in order, that you ask yourself as you start to factor a polynomial? What do you need to do as a result of the answer to each question?

214. On your paper draw the chart that summarizes the factoring strategy. Try to do it without looking at the book. When you are done, look back at the book to finish it or verify it.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
recognize a preliminary strategy to factor polynomials completely.			
factor trinomials of the form $ax^2 + bx + c$ with a GCF.			
factor trinomials using trial and error.			
factor trinomials using the "ac" method.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

7.4 Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes
- Choose method to factor a polynomial completely

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(12x)^2$.
   If you missed this problem, review **Example 6.23**.
2. Multiply: $(m+4)^2$.
   If you missed this problem, review **Example 6.47**.
3. Multiply: $(p-9)^2$.
   If you missed this problem, review **Example 6.48**.
4. Multiply: $(k+3)(k-3)$.
   If you missed this problem, review **Example 6.52**.

The strategy for factoring we developed in the last section will guide you as you factor most binomials, trinomials, and polynomials with more than three terms. We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. You can square a binomial by using FOIL, but using the Binomial Squares pattern you saw in a previous chapter saves you a step. Let's review the Binomial Squares pattern by squaring a binomial using FOIL.

$$(3x+4)^2$$
$$(3x+4)(3x+4)$$
$$\overset{F}{9x^2} + \overset{O}{12x} + \overset{I}{12x} + \overset{L}{16}$$
$$9x^2 + 24x + 16$$

The first term is the square of the first term of the binomial and the last term is the square of the last. The middle term is twice the product of the two terms of the binomial.

$$(3x)^2 + 2(3x \cdot 4) + 4^2$$
$$9x^2 + 24x + 16$$

The trinomial $9x^2$ + 24 +16 is called a perfect square trinomial. It is the square of the binomial $3x+4$.

We'll repeat the Binomial Squares Pattern here to use as a reference in factoring.

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad (a-b)^2 = a^2 - 2ab + b^2$$

When you square a binomial, the product is a perfect square trinomial. In this chapter, you are learning to factor—now, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form $ax^2 + bx + c$. But if you recognize that the first and last terms are squares and the trinomial fits the **perfect square trinomials pattern**, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If *a* and *b* are real numbers,

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad a^2 - 2ab + b^2 = (a - b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it twice the product, 2*ab*? If everything checks, you can easily write the factors.

EXAMPLE 7.42 HOW TO FACTOR PERFECT SQUARE TRINOMIALS

Factor: $9x^2 + 12x + 4$.

Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$?		
• Is the first term a perfect square? Write it as a square, a^2.	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$.	$9x^2 + 12x + 4$ $(3x)^2$
• Is the last term a perfect square? Write it as a square, b^2.	Is 4 a perfect square? Yes—write it as $(2)^2$.	$(3x)^2 \quad (2)^2$
• Check the middle term. Is it 2*ab*?	Is 12*x* twice the product of 3*x* and 2? Does it match? Yes, so we have a perfect square trinomial!	$(3x)^2 \quad (2)^2$ $2(3x)(2)$ $12x$
Step 2. Write the square of the binomial.	Write it as the square of a binomial.	$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check. $(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓		

TRY IT : : 7.83 Factor: $4x^2 + 12x + 9$.

TRY IT : : 7.84 Factor: $9y^2 + 24y + 16$.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

HOW TO : : FACTOR PERFECT SQUARE TRINOMIALS.

Step 1. Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
• Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
• Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
• Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3. Check by multiplying.		

We'll work one now where the middle term is negative.

EXAMPLE 7.43

Factor: $81y^2 - 72y + 16$.

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a - b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2 \qquad (4)^2$
Check the middle term.	$(9y)^2 \searrow 2(9y)(4) \swarrow (4)^2$ $72y$
Does is match $(a - b)^2$? Yes.	$\overset{a^2}{(9y)^2} \overset{-2}{-2} \cdot \overset{a}{9y} \cdot \overset{b}{4} \overset{+}{+} \overset{b^2}{4^2}$
Write the square of a binomial.	$(9y - 4)^2$
Check by mulitplying.	
$(9y - 4)^2$	
$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$	
$81y^2 - 72y + 16$ ✓	

TRY IT : : 7.85 Factor: $64y^2 - 80y + 25$.

TRY IT : : 7.86 Factor: $16z^2 - 72z + 81$.

The next example will be a perfect square trinomial with two variables.

EXAMPLE 7.44

Factor: $36x^2 + 84xy + 49y^2$.

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$\overset{a^2}{(6x)^2} + \overset{+2}{2} \cdot \overset{a}{6x} \cdot \overset{b}{7y} + \overset{b^2}{(7y)^2}$
Factor.	$(6x + 7y)^2$
Check by mulitplying.	
$(6x + 7y)^2$	
$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$	
$36x^2 + 84xy + 49y^2$ ✓	

TRY IT : : 7.87 Factor: $49x^2 + 84xy + 36y^2$.

TRY IT : : 7.88 Factor: $64m^2 + 112mn + 49n^2$.

EXAMPLE 7.45

Factor: $9x^2 + 50x + 25$.

Solution

	$9x^2 + 50x + 25$
Are the fir t and last terms perfect squares?	$(3x)^2 \quad (5)^2$
Check the middle term—is it $2ab$?	$(3x)^2 \searrow \underset{30x}{2(3x)(5)} \swarrow (5)^2$
No! $30x \neq 50x$	This does not fit he pattern!
Factor using the “ac” method.	$9x^2 + 50x + 25$

Notice: $\underset{225}{\overset{ac}{9 \cdot 25}}$ and $\begin{array}{l} 5 \cdot 45 = 225 \\ 5 + 45 = 50 \end{array}$

Split the middle term.	$9x^2 + 5x + 45x + 25$
Factor by grouping.	$x(9x + 5) + 5(9x + 5)$
	$(9x + 5)(x + 5)$

Check.

$(9x + 5)(x + 5)$

$9x^2 + 45x + 5x + 25$

$9x^2 + 50x + 25$ ✓

TRY IT : : 7.89 Factor: $16r^2 + 30rs + 9s^2$.

TRY IT : : 7.90 Factor: $9u^2 + 87u + 100$.

Remember the very first step in our Strategy for Factoring Polynomials? It was to ask "is there a greatest common factor?" and, if there was, you factor the GCF before going any further. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 7.46

Factor: $36x^2y - 48xy + 16y$.

Solution

	$36x^2y - 48xy + 16y$
Is there a GCF? Yes, 4*y*, so factor it out.	$4y\left(9x^2 - 12x + 4\right)$
Is this a perfect square trinomial?	
Verify the pattern.	$4y[\overset{a^2}{(3x)^2} \overset{-2}{-2} \cdot \overset{a}{3x} \cdot \overset{b}{2} \overset{+}{+} \overset{b^2}{2^2}]$
Factor.	$4y(3x - 2)^2$
Remember: Keep the factor 4*y* in the final product.	
Check.	
$4y(3x - 2)^2$	
$4y\left[(3x)^2 - 2 \cdot 3x \cdot 2 + 2^2\right]$	
$4y(9x)^2 - 12x + 4$	
$36x^2y - 48xy + 16y$ ✓	

TRY IT : : 7.91 Factor: $8x^2y - 24xy + 18y$.

TRY IT : : 7.92 Factor: $27p^2q + 90pq + 75q$.

Factor Differences of Squares

The other special product you saw in the previous was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$(3x - 4)(3x + 4)$$
$$9x^2 - 16$$

Remember, when you multiply conjugate binomials, the middle terms of the product add to 0. All you have left is a binomial, the difference of squares.

Multiplying conjugates is the only way to get a binomial from the product of two binomials.

Product of Conjugates Pattern

If *a* and *b* are real numbers

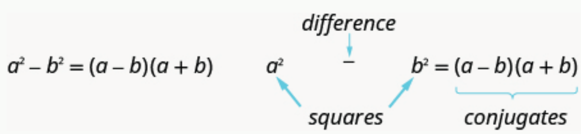
$$(a-b)(a+b)=a^2-b^2$$

The product is called a difference of squares.

To factor, we will use the product pattern “in reverse” to factor the difference of squares. A **difference of squares** factors to a product of conjugates.

Difference of Squares Pattern

If a and b are real numbers,



Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

EXAMPLE 7.47 HOW TO FACTOR DIFFERENCES OF SQUARES

Factor: x^2-4.

Solution

Step 1. Does the binomial fit the pattern? • Is this a difference? • Are the first and last terms perfect squares?	 Yes Yes	x^2-4 x^2-4
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 - b^2$ $(x)^2-2^2$
Step 3. Write the product of conjugates.		$(a-b)(a+b)$ $(x-2)(x+2)$
Step 4. Check.		$(x-2)(x+2)$ x^2-4 ✓

TRY IT : : 7.93 Factor: h^2-81.

TRY IT : : 7.94 Factor: k^2-121.

HOW TO : : FACTOR DIFFERENCES OF SQUARES.

Step 1. Does the binomial fit he pattern? a^2-b^2
- Is this a diffe ence? ____ − ____
- Are the fir t and last terms perfect squares?

Step 2. Write them as squares. $(a)^2-(b)^2$

Step 3. Write the product of conjugates. $(a-b)(a+b)$

Step 4. Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

Don't forget that 1 is a perfect square. We'll need to use that fact in the next example.

EXAMPLE 7.48

Factor: $64y^2 - 1$.

Solution

	$64y^2 - 1$
Is this a difference? Yes.	$64y^2 - 1$
Are the first and last terms perfect squares?	
Yes - write them as squares.	$\overset{a^2}{(8y)^2} - \overset{b^2}{1^2}$
Factor as the product of conjugates.	$\overset{(a-b)(a+b)}{(8y-1)(8y+1)}$
Check by multiplying.	
$(8y - 1)(8y + 1)$	
$64y^2 - 1$ ✓	

TRY IT : : 7.95 Factor: $m^2 - 1$.

TRY IT : : 7.96 Factor: $81y^2 - 1$.

EXAMPLE 7.49

Factor: $121x^2 - 49y^2$.

Solution

	$121x^2 - 49y^2$
Is this a diffe ence of squares? Yes.	$(11x)^2 - (7y)^2$
Factor as the product of conjugates.	$(11x - 7y)(11x + 7y)$
Check by multiplying.	

$(11x - 7y)(11x + 7y)$

$121x^2 - 49y^2$ ✓

TRY IT : : 7.97 Factor: $196m^2 - 25n^2$.

TRY IT : : 7.98 Factor: $144p^2 - 9q^2$.

The binomial in the next example may look “backwards,” but it’s still the difference of squares.

EXAMPLE 7.50

Factor: $100 - h^2$.

Solution

$$100 - h^2$$

Is this a diffe ence of squares? Yes. $(10)^2 - (h)^2$

Factor as the product of conjugates. $(10 - h)(10 + h)$

Check by multiplying.

$(10 - h)(10 + h)$
$100 - h^2$ ✓

Be careful not to rewrite the original expression as $h^2 - 100$.

Factor $h^2 - 100$ on your own and then notice how the result differs from $(10 - h)(10 + h)$.

TRY IT : : 7.99 Factor: $144 - x^2$.

TRY IT : : 7.100 Factor: $169 - p^2$.

To completely factor the binomial in the next example, we’ll factor a difference of squares twice!

EXAMPLE 7.51

Factor: $x^4 - y^4$.

Solution

$$x^4 - y^4$$

Is this a diffe ence of squares? Yes. $\left(x^2\right)^2 - \left(y^2\right)^2$

Factor it as the product of conjugates. $\left(x^2 - y^2\right)\left(x^2 + y^2\right)$

Notice the fir t binomial is also a diffe ence of squares! $\left((x)^2 - (y)^2\right)\left(x^2 + y^2\right)$

Factor it as the product of conjugates. The last factor, the sum of squares, cannot be factored. $(x - y)(x + y)\left(x^2 + y^2\right)$

Check by multiplying.

$(x - y)(x + y)\left(x^2 + y^2\right)$
$[(x - y)(x + y)]\left(x^2 + y^2\right)$
$\left(x^2 - y^2\right)\left(x^2 + y^2\right)$
$x^4 - y^4$ ✓

TRY IT : : 7.101 Factor: $a^4 - b^4$.

TRY IT : : 7.102 Factor: $x^4 - 16$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and you won't recognize the perfect squares until you factor the GCF.

EXAMPLE 7.52

Factor: $8x^2y - 18y$.

Solution

	$8x^2y - 98y$
Is there a GCF? Yes, $2y$—factor it out!	$2y(4x^2 - 49)$
Is the binomial a diffe ence of squares? Yes.	$2y((2x)^2 - (7)^2)$
Factor as a product of conjugates.	$2y(2x - 7)(2x + 7)$

Check by multiplying.

$$2y(2x - 7)(2x + 7)$$
$$2y[(2x - 7)(2x + 7)]$$
$$2y(4x^2 - 49)$$
$$8x^2y - 98y \checkmark$$

TRY IT : : 7.103 Factor: $7xy^2 - 175x$.

TRY IT : : 7.104 Factor: $45a^2b - 80b$.

EXAMPLE 7.53

Factor: $6x^2 + 96$.

Solution

	$6x^2 + 96$
Is there a GCF? Yes, 6—factor it out!	$6(x^2 + 16)$
Is the binomial a diffe ence of squares? No, it is a sum of squares. Sums of squares do not factor!	

Check by multiplying.

$$6(x^2 + 16)$$
$$6x^2 + 96 \checkmark$$

TRY IT : : 7.105 Factor: $8a^2 + 200$.

TRY IT : : 7.106 Factor: $36y^2 + 81$.

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

We'll check the first pattern and leave the second to you.

	$(a+b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

Sum and Difference of Cubes Pattern

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

The two patterns look very similar, don't they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

same sign

opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

same sign

opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It can be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Figure 7.3.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Figure 7.3

EXAMPLE 7.54 HOW TO FACTOR THE SUM OR DIFFERENCE OF CUBES

Factor: $x^3 + 64$.

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	 This is a sum. Yes	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3	$\overset{a^3}{x^3} + \overset{b^3}{4^3}$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\left(\overset{a}{x} + \overset{b}{4}\right)\left(\overset{a^2}{x^2} - \overset{ab}{4x} + \overset{b^2}{4^2}\right)$
Step 4. Simplify inside the parentheses.	Simplify 4^2.	$(x + 4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$\begin{array}{r} x^2 - 4x + 16 \\ x + 4 \\ \hline 4x^2 - 16x + 64 \\ x^3 - 4x^2 + 16x \\ \hline x^3 + 64 \checkmark \end{array}$

TRY IT : : 7.107 Factor: $x^3 + 27$.

TRY IT : : 7.108 Factor: $y^3 + 8$.

HOW TO : : FACTOR THE SUM OR DIFFERENCE OF CUBES.

To factor the sum or difference of cubes:

Step 1. Does the binomial fit the sum or difference of cubes pattern?
- Is it a sum or difference?
- Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses

Step 5. Check by multiplying the factors.

EXAMPLE 7.55

Factor: $x^3 - 1000$.

Solution

	$x^3 - 1000$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{x^3} - \overset{b^3}{10^3}$

Use the difference of cubes pattern.	$\overset{a\ -\ b}{(x-10)}\ \overset{a^2\ +\ ab\ +\ b^2}{(x^2+10\cdot x+10^2)}$
Simplify.	$\overset{a\ -\ b}{(x-10)}\ \overset{a^2\ +\ ab\ +\ b^2}{(x^2+10x+100)}$
Check by multiplying.	

$(x-10)\ (x^2+10x+100)$

$$\begin{array}{r} x^2+10x+100 \\ x-100 \\ \hline x^3+10x^2+100x \\ -10x^2-100x-1000\ ✓ \\ \hline x^3 \qquad\qquad -1000 \end{array}$$

> TRY IT : : 7.109 Factor: u^3-125.

> TRY IT : : 7.110 Factor: v^3-343.

Be careful to use the correct signs in the factors of the sum and difference of cubes.

EXAMPLE 7.56

Factor: $512-125p^3$.

Solution

	$512-125p^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3\ -\ b^3}{8^3-(5p)^3}$
Use the difference of cubes pattern.	$\overset{a\ -\ b}{(8-5p)}\ \overset{a^2\ +\ ab\ +\ b^2}{(8^2+8\cdot 5p+(5p)^2)}$
Simplify.	$\overset{a\ -\ b}{(8-5p)}\ \overset{a^2\ +\ ab\ +\ b^2}{(64+40p+25p^2)}$
Check by multiplying.	We'll leave the check to you.

> TRY IT : : 7.111 Factor: $64-27x^3$.

> TRY IT : : 7.112 Factor: $27-8y^3$.

EXAMPLE 7.57

Factor: $27u^3-125v^3$.

Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} - \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$(\overset{a}{3u} - \overset{b}{5v})(\overset{a^2}{(3u)^2} + \overset{ab}{3u \cdot 5v} + \overset{b^2}{(5v)^2})$
Simplify.	$(\overset{a}{3u} - \overset{b}{5v})(\overset{a^2}{9u^2} + \overset{ab}{15uv} + \overset{b^2}{25v^2})$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 7.113 Factor: $8x^3 - 27y^3$.

TRY IT : : 7.114 Factor: $1000m^3 - 125n^3$.

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

EXAMPLE 7.58

Factor: $5m^3 + 40n^3$.

Solution

	$5m^3 + 40n^3$
Factor the common factor.	$5(m^3 + 8n^3)$
This binomial is a sum. The first and last terms are perfect cubes.	
Write the terms as cubes.	$5(\overset{a^3}{m^3} + \overset{b^3}{(2n)^3})$
Use the sum of cubes pattern.	$5(\overset{a}{m} + \overset{b}{2n})(\overset{a^2}{m^2} - \overset{ab}{m \cdot 2n} + \overset{b^2}{(2n)^2})$
Simplify.	$5(\overset{a}{m} + \overset{b}{2n})(\overset{a^2}{m^2} - \overset{ab}{2mn} + \overset{b^2}{4n^2})$

Check. To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by 5. We'll leave the multiplication for you.

$$5(m + 2n)\left(m^2 - 2mn + 4n^2\right)$$

TRY IT : : 7.115 Factor: $500p^3 + 4q^3$.

TRY IT : : 7.116 Factor: $432c^3 + 686d^3$.

MEDIA : :

Access these online resources for additional instruction and practice with factoring special products.

- **Sum of Difference of Cubes (https://openstax.org/l/25SumCubes)**
- **Difference of Cubes Factoring (https://openstax.org/l/25DiffCubes)**

7.4 EXERCISES

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor.

215. $16y^2 + 24y + 9$

216. $25v^2 + 20v + 4$

217. $36s^2 + 84s + 49$

218. $49s^2 + 154s + 121$

219. $100x^2 - 20x + 1$

220. $64z^2 - 16z + 1$

221. $25n^2 - 120n + 144$

222. $4p^2 - 52p + 169$

223. $49x^2 - 28xy + 4y^2$

224. $25r^2 - 60rs + 36s^2$

225. $25n^2 + 25n + 4$

226. $100y^2 - 52y + 1$

227. $64m^2 - 34m + 1$

228. $100x^2 - 25x + 1$

229. $10k^2 + 80k + 160$

230. $64x^2 - 96x + 36$

231. $75u^3 - 30u^2v + 3uv^2$

232. $90p^3 + 300p^2q + 250pq^2$

Factor Differences of Squares

In the following exercises, factor.

233. $x^2 - 16$

234. $n^2 - 9$

235. $25v^2 - 1$

236. $169q^2 - 1$

237. $121x^2 - 144y^2$

238. $49x^2 - 81y^2$

239. $169c^2 - 36d^2$

240. $36p^2 - 49q^2$

241. $4 - 49x^2$

242. $121 - 25s^2$

243. $16z^4 - 1$

244. $m^4 - n^4$

245. $5q^2 - 45$

246. $98r^3 - 72r$

247. $24p^2 + 54$

248. $20b^2 + 140$

Factor Sums and Differences of Cubes

In the following exercises, factor.

249. $x^3 + 125$

250. $n^3 + 512$

251. $z^3 - 27$

252. $v^3 - 216$

253. $8 - 343t^3$

254. $125 - 27w^3$

255. $8y^3 - 125z^3$

256. $27x^3 - 64y^3$

257. $7k^3 + 56$

258. $6x^3 - 48y^3$

259. $2 - 16y^3$

260. $-2x^3 - 16y^3$

Mixed Practice

In the following exercises, factor.

261. $64a^2 - 25$

262. $121x^2 - 144$

263. $27q^2 - 3$

264. $4p^2 - 100$

265. $16x^2 - 72x + 81$

266. $36y^2 + 12y + 1$

267. $8p^2 + 2$

268. $81x^2 + 169$

269. $125 - 8y^3$

270. $27u^3 + 1000$

271. $45n^2 + 60n + 20$

272. $48q^3 - 24q^2 + 3q$

Everyday Math

273. Landscaping Sue and Alan are planning to put a 15 foot square swimming pool in their backyard. They will surround the pool with a tiled deck, the same width on all sides. If the width of the deck is *w*, the total area of the pool and deck is given by the trinomial $4w^2 + 60w + 225$. Factor the trinomial.

274. Home repair The height a twelve foot ladder can reach up the side of a building if the ladder's base is *b* feet from the building is the square root of the binomial $144 - b^2$. Factor the binomial.

Writing Exercises

275. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

276. How do you recognize the binomial squares pattern?

277. Explain why $n^2 + 25 \neq (n + 5)^2$. Use algebra, words, or pictures.

278. Maribel factored $y^2 - 30y + 81$ as $(y - 9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

7.5 General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor $y^2 - 2y - 24$.
   If you missed this problem, review **Example 7.23**.
2. Factor $3t^2 + 17t + 10$.
   If you missed this problem, review **Example 7.38**.
3. Factor $36p^2 - 60p + 25$.
   If you missed this problem, review **Example 7.42**.
4. Factor $5x^2 - 80$.
   If you missed this problem, review **Example 7.52**.

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. (In your next algebra course, more methods will be added to your repertoire.) The figure below summarizes all the factoring methods we have covered. **Factor polynomials.** outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

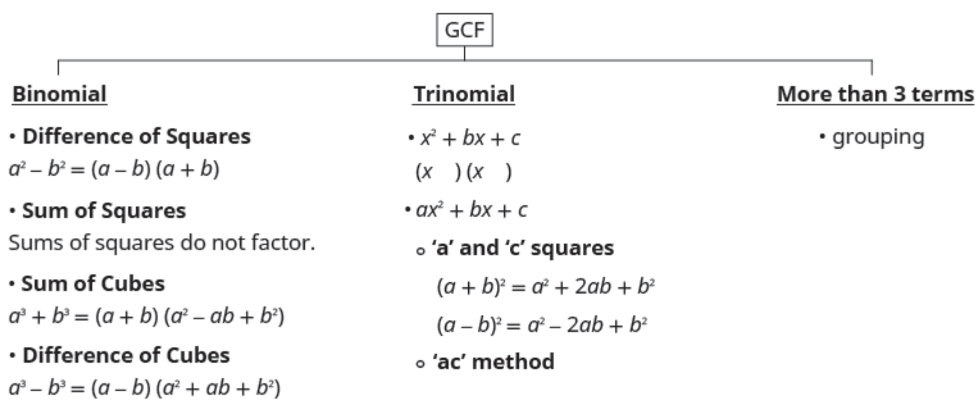


Figure 7.4

HOW TO : : FACTOR POLYNOMIALS.

Step 1. Is there a greatest common factor?
- Factor it out.

Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
- If it is a binomial:
  Is it a sum?
  - Of squares? Sums of squares do not factor.
  - Of cubes? Use the sum of cubes pattern.

  Is it a difference?
  - Of squares? Factor as the product of conjugates.
  - Of cubes? Use the difference of cubes pattern.
- If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
  - If a and c are squares, check if it fits the trinomial square pattern.
  - Use the trial and error or "ac" method.
- If it has more than three terms:
  Use the grouping method.

Step 3. Check.
- Is it factored completely?
- Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are prime!

EXAMPLE 7.59

Factor completely: $4x^5 + 12x^4$.

Solution

Is there a GCF?	Yes, $4x^4$.	$4x^5 + 12x^4$
	Factor out the GCF.	$4x^4(x+3)$
In the parentheses, is it a binomial, a trinomial, or are there more than three terms?	Binomial.	
Is it a sum?		Yes.
Of squares? Of cubes?		No.
Check.		
Is the expression factored completely?		Yes.
Multiply.		

$$4x^4(x+3)$$
$$4x^4 \cdot x + 4x^4 \cdot 3$$
$$4x^5 + 12x^4 \checkmark$$

TRY IT : : 7.117 Factor completely: $3a^4 + 18a^3$.

TRY IT : : 7.118 Factor completely: $45b^6 + 27b^5$.

EXAMPLE 7.60

Factor completely: $12x^2 - 11x + 2$.

Solution

	$12x^2 - 11x + 2$
Is there a GCF?	No.
Is it a binomial, trinomial, or are there more than three terms?	Trinomial.
Are a and c perfect squares?	No, $a = 12$, not a perfect square.
Use trial and error or the "ac" method. We will use trial and error here.	$12x^2 - 11x + 2$ $1x, 12x$ $-1, -2$ $2x, 6x$ $3x, 4x$

$12x^2 - 11x + 2$	
Possible factors	**Product**
$(x - 1)(12x - 2)$	Not an option
$(x - 2)(12x - 1)$	$12x^2 - 25x + 2$
$(2x - 1)(6x - 2)$	Not an option
$(2x - 2)(6x - 1)$	Not an option
$(3x - 1)(4x - 2)$	Not an option
$(3x - 2)(4x - 1)$	$12x^2 - 11x + 2$

If the trinomial has no common factors, then neither factor can contain a common factor. That means each of these combinations is not an option.

Check.

$$(3x - 2)(4x - 1)$$

$$12x^2 - 3x - 8x + 2$$

$$12x^2 - 11x + 2 \checkmark$$

TRY IT : : 7.119 Factor completely: $10a^2 - 17a + 6$.

TRY IT : : 7.120 Factor completely: $8x^2 - 18x + 9$.

EXAMPLE 7.61

Factor completely: $g^3 + 25g$.

Solution

Is there a GCF?	Yes, g.	$g^3 + 25g$
Factor out the GCF.		$g(g^2 + 25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum ? Of squares?	Yes.	Sums of squares are prime.

Check.

Is the expression factored completely? Yes.

Multiply.

$g(g^2 + 25)$

$g^3 + 25g$ ✓

TRY IT : : 7.121 Factor completely: $x^3 + 36x$.

TRY IT : : 7.122 Factor completely: $27y^2 + 48$.

EXAMPLE 7.62

Factor completely: $12y^2 - 75$.

Solution

Is there a GCF?	Yes, 3.	$12y^2 - 75$
Factor out the GCF.		$3(4y^2 - 25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum?	No.	
Is it a diffe ence? Of squares or cubes?	Yes, squares.	$3((2y)^2 - (5)^2)$
Write as a product of conjugates.		$3(2y - 5)(2y + 5)$

Check.

Is the expression factored completely? Yes.

Neither binomial is a diffe ence of squares.

Multiply.

$3(2y - 5)(2y + 5)$

$3(4y^2 - 25)$

$12y^2 - 75$ ✓

TRY IT : : 7.123 Factor completely: $16x^3 - 36x$.

TRY IT : : 7.124 Factor completely: $27y^2 - 48$.

EXAMPLE 7.63

Factor completely: $4a^2 - 12ab + 9b^2$.

Solution

Is there a GCF?	No.	$4a^2 - 12ab + 9b^2$
Is it a binomial, trinomial, or are there more terms?		
Trinomial with $a \neq 1$. But the first term is a perfect square.		
Is the last term a perfect square?	Yes.	$(2a)^2 - 12ab + (3b)^2$
Does it fit the pattern, $a^2 - 2ab + b^2$?	Yes.	$(2a)^2 - 12ab + (3b)^2$; $-2(2a)(3b)$ = $12ab$
Write it as a square.		$(2a - 3b)^2$
Check your answer.		
Is the expression factored completely?		
Yes.		
The binomial is not a difference of squares.		
Multiply.		
$(2a - 3b)^2$		
$(2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2$		
$4a^2 - 12ab + 9b^2$ ✓		

TRY IT : : 7.125 Factor completely: $4x^2 + 20xy + 25y^2$.

TRY IT : : 7.126 Factor completely: $9m^2 + 42mn + 49n^2$.

EXAMPLE 7.64

Factor completely: $6y^2 - 18y - 60$.

Solution

Is there a GCF?	Yes, 6.	$6y^2 - 18y - 60$
Factor out the GCF.	Trinomial with leading coefficient	$6(y^2 - 3y - 10)$
In the parentheses, is it a binomial, trinomial, or are there more terms?		
"Undo" FOIL.	$6(y \quad)(y \quad)$	$6(y + 2)(y - 5)$
Check your answer.		
Is the expression factored completely?		Yes.
Neither binomial is a diffe ence of squares.		
Multiply.		

$$6(y + 2)(y - 5)$$
$$6(y^2 - 5y + 2y - 10)$$
$$6(y^2 - 3y - 10)$$
$$6y^2 - 18y - 60 \checkmark$$

> **TRY IT : :** 7.127 Factor completely: $8y^2 + 16y - 24$.

> **TRY IT : :** 7.128 Factor completely: $5u^2 - 15u - 270$.

EXAMPLE 7.65

Factor completely: $24x^3 + 81$.

Solution

Is there a GCF?	Yes, 3.	$24x^3 + 81$
Factor it out.		$3(8x^3 + 27)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or difference?	Sum.	
Of squares or cubes?	Sum of cubes.	$3(\overset{a^3}{(2x)^3} \overset{+}{+} \overset{b^3}{(3)^3})$
Write it using the sum of cubes pattern.		$3(\overset{a}{2x} \overset{+}{+} \overset{b}{3})(\overset{a^2}{(2x)^2} \overset{-}{-} \overset{ab}{2x \cdot 3} \overset{+}{+} \overset{b^2}{3^2})$
Is the expression factored completely?	Yes.	$3(2x + 3)(4x^2 - 6x + 9)$
Check by multiplying.		We leave the check to you.

> **TRY IT : :** 7.129 Factor completely: $250m^3 + 432$.

TRY IT : : 7.130 Factor completely: $81q^3 + 192$.

EXAMPLE 7.66

Factor completely: $2x^4 - 32$.

Solution

Is there a GCF?	Yes, 2.	$2x^4 - 32$
Factor it out.		$2(x^4 - 16)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Binomial.	
Is it a sum or diffe ence?	Yes.	
Of squares or cubes?	Diffe ence of squares.	$2\left(\left(x^2\right)^2 - (4)^2\right)$
Write it as a product of conjugates.		$2(x^2 - 4)(x^2 + 4)$
The fir t binomial is again a diffe ence of squares.		$2((x)^2 - (2)^2)(x^2 + 4)$
Write it as a product of conjugates.		$2(x - 2)(x + 2)(x^2 + 4)$
Is the expression factored completely?	Yes.	
None of these binomials is a diffe ence of squares.		

Check your answer.

Multiply.

$2(x - 2)(x + 2)(x^2 + 4)$

$2(x^2 - 4)(x^2 + 4)$

$2(x^4 - 16)$

$2x^4 - 32$ ✓

TRY IT : : 7.131 Factor completely: $4a^4 - 64$.

TRY IT : : 7.132 Factor completely: $7y^4 - 7$.

EXAMPLE 7.67

Factor completely: $3x^2 + 6bx - 3ax - 6ab$.

Solution

Is there a GCF?	Yes, 3.	$3x^2 + 6bx - 3ax - 6ab$
Factor out the GCF.		$3\left(x^2 + 2bx - ax - 2ab\right)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	More than 3 terms.	
Use grouping.		$3[x(x + 2b) - a(x + 2b)]$ $3(x + 2b)(x - a)$

Check your answer.

Is the expression factored completely? Yes.
Multiply.

$3(x + 2b)(x - a)$
$3\left(x^2 - ax + 2bx - 2ab\right)$
$3x^2 - 3ax + 6bx - 6ab$ ✓

TRY IT : : 7.133 Factor completely: $6x^2 - 12xc + 6bx - 12bc$.

TRY IT : : 7.134 Factor completely: $16x^2 + 24xy - 4x - 6y$.

EXAMPLE 7.68

Factor completely: $10x^2 - 34x - 24$.

Solution

Is there a GCF?	Yes, 2.	$10x^2 - 34x - 24$
Factor out the GCF.		$2\left(5x^2 - 17x - 12\right)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms?	Trinomial with $a \neq 1$.	
Use trial and error or the "ac" method.		$2(5x^2 - 17x - 12)$ $2(5x + 3)(x - 4)$

Check your answer. Is the expression factored completely? Yes.

Multiply.

$2(5x + 3)(x - 4)$
$2\left(5x^2 - 20x + 3x - 12\right)$
$2\left(5x^2 - 17x - 12\right)$
$10x^2 - 34x - 24$ ✓

TRY IT : : 7.135 Factor completely: $4p^2 - 16p + 12$.

TRY IT : : 7.136 Factor completely: $6q^2 - 9q - 6$.

7.5 EXERCISES

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

279. $10x^4 + 35x^3$

280. $18p^6 + 24p^3$

281. $y^2 + 10y - 39$

282. $b^2 - 17b + 60$

283. $2n^2 + 13n - 7$

284. $8x^2 - 9x - 3$

285. $a^5 + 9a^3$

286. $75m^3 + 12m$

287. $121r^2 - s^2$

288. $49b^2 - 36a^2$

289. $8m^2 - 32$

290. $36q^2 - 100$

291. $25w^2 - 60w + 36$

292. $49b^2 - 112b + 64$

293. $m^2 + 14mn + 49n^2$

294. $64x^2 + 16xy + y^2$

295. $7b^2 + 7b - 42$

296. $3n^2 + 30n + 72$

297. $3x^3 - 81$

298. $5t^3 - 40$

299. $k^4 - 16$

300. $m^4 - 81$

301. $15pq - 15p + 12q - 12$

302. $12ab - 6a + 10b - 5$

303. $4x^2 + 40x + 84$

304. $5q^2 - 15q - 90$

305. $u^5 + u^2$

306. $5n^3 + 320$

307. $4c^2 + 20cd + 81d^2$

308. $25x^2 + 35xy + 49y^2$

309. $10m^4 - 6250$

310. $3v^4 - 768$

Everyday Math

311. Watermelon drop A springtime tradition at the University of California San Diego is the Watermelon Drop, where a watermelon is dropped from the seventh story of Urey Hall.

ⓐ The binomial $-16t^2 + 80$ gives the height of the watermelon t seconds after it is dropped. Factor the greatest common factor from this binomial.

ⓑ If the watermelon is thrown down with initial velocity 8 feet per second, its height after t seconds is given by the trinomial $-16t^2 - 8t + 80$. Completely factor this trinomial.

312. Pumpkin drop A fall tradition at the University of California San Diego is the Pumpkin Drop, where a pumpkin is dropped from the eleventh story of Tioga Hall.

ⓐ The binomial $-16t^2 + 128$ gives the height of the pumpkin t seconds after it is dropped. Factor the greatest common factor from this binomial.

ⓑ If the pumpkin is thrown down with initial velocity 32 feet per second, its height after t seconds is given by the trinomial $-16t^2 - 32t + 128$. Completely factor this trinomial.

Writing Exercises

313. The difference of squares $y^4 - 625$ can be factored as $(y^2 - 25)(y^2 + 25)$. But it is not *completely* factored. What more must be done to completely factor it?

314. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, 'ac' method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

7.6 Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations by using the Zero Product Property
- Solve quadratic equations factoring
- Solve applications modeled by quadratic equations

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
   If you missed this problem, review **Example 2.27**.
2. Solve: $10a=0$.
   If you missed this problem, review **Example 2.13**.
3. Combine like terms: $12x^2-6x+4x$.
   If you missed this problem, review **Example 1.24**.
4. Factor n^3-9n^2-22n completely.
   If you missed this problem, review **Example 7.32**.

We have already solved linear equations, equations of the form $ax+by=c$. In linear equations, the variables have no exponents. Quadratic equations are equations in which the variable is squared. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0 \qquad 3y^2+4y=10 \qquad 64u^2-81=0 \qquad n(n+1)=42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n.

The general form of a quadratic equation is $ax^2+bx+c=0, \text{ with } a\neq 0$.

Quadratic Equation

An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a,\ b,\ \text{ and } c \text{ are real numbers and } a\neq 0$$

To solve quadratic equations we need methods different than the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Solve Quadratic Equations Using the Zero Product Property

We will first solve some quadratic equations by using the Zero Product Property. The **Zero Product Property** says that if the product of two quantities is zero, it must be that at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $a\cdot b=0$, then either $a=0$ or $b=0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

EXAMPLE 7.69 HOW TO USE THE ZERO PRODUCT PROPERTY TO SOLVE A QUADRATIC EQUATION

Solve: $(x+1)(x-4)=0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(x+1)(x-4)=0$ $x+1=0$ or $x-4=0$
Step 2. Solve the linear equations.	Solve each equation.	$x=-1$ or $x=4$
Step 3. Check.	Substitute each solution separately into the original equation.	$x=-1$ $(x+1)(x-4)=0$ $(-1+1)(-1-4)\stackrel{?}{=}0$ $(0)(-5)\stackrel{?}{=}0$ $0=0$ ✓ $x=4$ $(x+1)(x-4)=0$ $(4+1)(4-4)\stackrel{?}{=}0$ $(5)(0)\stackrel{?}{=}0$ $0=0$ ✓

TRY IT : : 7.137 Solve: $(x-3)(x+5)=0$.

TRY IT : : 7.138 Solve: $(y-6)(y+9)=0$.

We usually will do a little more work than we did in this last example to solve the linear equations that result from using the Zero Product Property.

EXAMPLE 7.70

Solve: $(5n-2)(6n-1)=0$.

Solution

		$(5n-2)(6n-1)=0$
Use the Zero Product Property to set each factor to 0.	$5n-2=0$	$6n-1=0$
Solve the equations.	$n=\frac{2}{5}$	$n=\frac{1}{6}$
Check your answers.		

$n = \frac{2}{5}$	$n = \frac{1}{6}$
$(5n - 2)(6n - 1) = 0$	$(5n - 2)(6n - 1) = 0$
$\left(5 \cdot \frac{2}{5} - 2\right)\left(6 \cdot \frac{2}{5} - 1\right) \stackrel{?}{=} 0$	$\left(5 \cdot \frac{1}{6} - 2\right)\left(6 \cdot \frac{1}{6} - 1\right) \stackrel{?}{=} 0$
$(2 - 2)\left(\frac{12}{5} - \frac{5}{5}\right) \stackrel{?}{=} 0$	$\left(\frac{5}{6} - \frac{12}{6}\right)(1 - 1) \stackrel{?}{=} 0$
$(0)\left(\frac{7}{5}\right) \stackrel{?}{=} 0$	$\left(-\frac{7}{6}\right)(0) \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

TRY IT : : 7.139 Solve: $(3m - 2)(2m + 1) = 0$.

TRY IT : : 7.140 Solve: $(4p + 3)(4p - 3) = 0$.

Notice when we checked the solutions that each of them made just one factor equal to zero. But the product was zero for both solutions.

EXAMPLE 7.71

Solve: $3p(10p + 7) = 0$.

Solution

	$3p(10p + 7) = 0$	
Use the Zero Product Property to set each factor to 0.	$3p = 0$	$10p + 7 = 0$
Solve the equations.	$p = 0$	$10p = -7$
		$p = -\frac{7}{10}$
Check your answers.		

$p = 0$	$p = -\frac{7}{10}$
$3p(10p + 7) = 0$	$3p(10p + 7) = 0$
$3 \cdot 0(10 \cdot 0 + 7) \stackrel{?}{=} 0$	$3\left(-\frac{7}{10}\right)10\left(-\frac{7}{10}\right) + 7 \stackrel{?}{=} 0$
$0(0 + 7) \stackrel{?}{=} 0$	$\left(-\frac{21}{10}\right)(-7 + 7) \stackrel{?}{=} 0$
$0(7) \stackrel{?}{=} 0$	$\left(-\frac{21}{10}\right)(0) \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

TRY IT : : 7.141 Solve: $2u(5u - 1) = 0$.

TRY IT : : 7.142 Solve: $w(2w+3)=0$.

It may appear that there is only one factor in the next example. Remember, however, that $(y-8)^2$ means $(y-8)(y-8)$.

EXAMPLE 7.72

Solve: $(y-8)^2=0$.

Solution

	$(y-8)^2=0$
Rewrite the left side as a product.	$(y-8)(y-8)=0$
Use the Zero Product Property and set each factor to 0.	$y-8=0 \quad y-8=0$
Solve the equations.	$y=8 \quad y=8$
When a solution repeats, we call it a double root.	
Check your answer.	

$y=8$
$(y-8)^2=0$
$(8-8)^2 \stackrel{?}{=} 0$
$(0)^2 \stackrel{?}{=} 0$
$0=0$ ✓

TRY IT : : 7.143 Solve: $(x+1)^2=0$.

TRY IT : : 7.144 Solve: $(v-2)^2=0$.

Solve Quadratic Equations by Factoring

Each of the equations we have solved in this section so far had one side in factored form. In order to use the Zero Product Property, the quadratic equation must be factored, with zero on one side. So we be sure to start with the quadratic equation in standard form, $ax^2+bx+c=0$. Then we factor the expression on the left.

EXAMPLE 7.73 HOW TO SOLVE A QUADRATIC EQUATION BY FACTORING

Solve: $x^2+2x-8=0$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	The equation is already in standard form.	$x^2+2x-8=0$
Step 2. Factor the quadratic expression.	Factor x^2+2x-8 $(x+4)(x-2)$	$(x+4)(x-2)=0$

Step 3. Use the Zero Product Property.	Set each factor equal to zero.	$x+4=0$ or $x-2=0$
Step 4. Solve the linear equations.	We have two linear equations.	$x=-4$ or $x=2$
Step 5. Check.	Substitute each solution separately into the original equation.	$x^2+2x-8=0$ $x=-4$ $(-4)^2-2(-4)-8\stackrel{?}{=}0$ $16+(-8)-8\stackrel{?}{=}0$ $0=0$ ✓ $x^2+2x-8=0$ $x=2$ $2^2-2(2)-8\stackrel{?}{=}0$ $4+4-8\stackrel{?}{=}0$ $0=0$ ✓

TRY IT : : 7.145 Solve: $x^2-x-12=0$.

TRY IT : : 7.146 Solve: $b^2+9b+14=0$.

HOW TO : : SOLVE A QUADRATIC EQUATION BY FACTORING.

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.

Step 2. Factor the quadratic expression.

Step 3. Use the Zero Product Property.

Step 4. Solve the linear equations.

Step 5. Check.

Before we factor, we must make sure the quadratic equation is in standard form.

EXAMPLE 7.74

Solve: $2y^2=13y+45$.

Solution

	$2y^2=13y+45$	
Write the quadratic equation in standard form.	$2y^2-13y-45=0$	
Factor the quadratic expression.	$(2y+5)(y-9)=0$	
Use the Zero Product Property to set each factor to 0.	$2y+5=0$	$y-9=0$
Solve each equation.	$y=-\frac{5}{2}$	$y=9$

Check your answers.

$y = -\frac{5}{2}$	$y = 9$
$2y^2 = 13y + 45$	$2y^2 = 13y + 45$
$2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$	$2(9)^2 \stackrel{?}{=} 13(9) + 45$
$2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$	$2(81) \stackrel{?}{=} 117 + 45$
$\frac{25}{2} = \frac{25}{2}$ ✓	$162 = 162$ ✓

TRY IT : : 7.147 Solve: $3c^2 = 10c - 8$.

TRY IT : : 7.148 Solve: $2d^2 - 5d = 3$.

EXAMPLE 7.75

Solve: $5x^2 - 13x = 7x$.

Solution

	$5x^2 - 13x = 7x$	
Write the quadratic equation in standard form.	$5x^2 - 20x = 0$	
Factor the left side of the equation.	$5x(x - 4) = 0$	
Use the Zero Product Property to set each factor to 0.	$5x = 0$	$x - 4 = 0$
Solve each equation.	$x = 0$	$x = 4$
Check your answers.		

$x = 0$	$x = 4$
$5x^2 - 13x = 7x$	$5x^2 - 13x = 7x$
$5(0)^2 - 13(0) \stackrel{?}{=} 7(0)$	$5(4)^2 - 13(4) \stackrel{?}{=} 7(4)$
$0 - 0 \stackrel{?}{=} 0$	$5(16) - 52 \stackrel{?}{=} 28$
$0 = 0$ ✓	$28 = 28$ ✓

TRY IT : : 7.149 Solve: $6a^2 + 9a = 3a$.

TRY IT : : 7.150 Solve: $45b^2 - 2b = -17b$.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

EXAMPLE 7.76

Solve: $144q^2 = 25$.

Solution

	$144q^2 = 25$
Write the quadratic equation in standard form.	$144q^2 - 25 = 0$
Factor. It is a diffe ence of squares.	$(12q - 5)(12q + 5) = 0$
Use the Zero Product Property to set each factor to 0.	$12q - 5 = 0 \qquad 12q + 5 = 0$
Solve each equation.	$12q = 5 \qquad 12q = -5$ $q = \frac{5}{12} \qquad q = -\frac{5}{12}$
Check your answers.	

TRY IT : : 7.151 Solve: $25p^2 = 49$.

TRY IT : : 7.152 Solve: $36x^2 = 121$.

The left side in the next example is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

EXAMPLE 7.77

Solve: $(3x - 8)(x - 1) = 3x$.

Solution

	$(3x - 8)(x - 1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x - 2)(x - 4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x - 2 = 0 \qquad x - 4 = 0$
Solve each equation.	$3x = 2 \qquad x = 4$ $x = \frac{2}{3}$
Check your answers.	The check is left to you!

TRY IT : : 7.153 Solve: $(2m + 1)(m + 3) = 12m$.

TRY IT : : 7.154 Solve: $(k+1)(k-1)=8$.

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree more than two by using the Zero Product Property, just like we solved quadratic equations.

EXAMPLE 7.78

Solve: $9m^3+100m=60m^2$.

Solution

	$9m^3+100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m = 0$
Factor the greatest common factor fir t.	$m\left(9m^2-60m+100\right) = 0$
Factor the trinomial.	$m(3m-10)(3m-10) = 0$
Use the Zero Product Property to set each factor to 0.	$m = 0 \quad 3m-10 = 0 \quad 3m-10 = 0$
Solve each equation.	$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

TRY IT : : 7.155 Solve: $8x^3=24x^2-18x$.

TRY IT : : 7.156 Solve: $16y^2=32y^3+2y$.

When we factor the quadratic equation in the next example we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

EXAMPLE 7.79

Solve: $4x^2=16x+84$.

Solution

	$4x^2 = 16x+84$
Write the quadratic equation in standard form.	$4x^2-16x-84 = 0$
Factor the greatest common factor fir t.	$4\left(x^2-4x-21\right) = 0$
Factor the trinomial.	$4(x-7)(x+3) = 0$
Use the Zero Product Property to set each factor to 0.	$4 \neq 0 \quad x-7 = 0 \quad x+3 = 0$
Solve each equation.	$4 \neq 0 \quad x = 7 \quad x = -3$
Check your answers.	The check is left to you.

TRY IT : : 7.157 Solve: $18a^2-30=-33a$.

TRY IT : : 7.158 Solve: $123b = -6 - 60b^2$.

Solve Applications Modeled by Quadratic Equations

The problem solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to quadratic equations. We will copy the problem solving strategy here so we can use it for reference.

HOW TO : : USE A PROBLEM-SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
Step 5. **Solve** the equation using good algebra techniques.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a quadratic equation.

EXAMPLE 7.80

The product of two consecutive integers is 132 . Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the fir t integer $n + 1 =$ the next consecutive integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive integers is 132. The fir t integer times the next integer is 132.
Translate to an equation.	$n(n+1) = 132$
Step 5. Solve the equation.	$n^2 + n = 132$
Bring all the terms to one side.	$n^2 + n - 132 = 0$
Factor the trinomial.	$(n-11)(n+12) = 0$
Use the zero product property.	$n - 11 = 0 \qquad n + 12 = 0$
Solve the equations.	$n = 11 \qquad n = -12$

There are two values for n that are solutions to this problem. So there are two sets of consecutive integers that will work.

If the fir t integer is $n = 11$	If the fir t integer is $n = -12$
then the next integer is $n + 1$	then the next integer is $n + 1$
$11 + 1$	$-12 + 1$
12	-11

Step 6. Check the answer.

The consecutive integers are $11, 12$ and $-11, -12$. The product $11 \cdot 12 = 132$ and the product $-11(-12) = 132$. Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are $11, 12$ and $-11, -12$.

TRY IT : : 7.159 The product of two consecutive integers is 240. Find the integers.

TRY IT : : 7.160 The product of two consecutive integers is 420. Find the integers.

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give 132.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

EXAMPLE 7.81

A rectangular garden has an area 15 square feet. The length of the garden is two feet more than the width. Find the length and width of the garden.

Solution

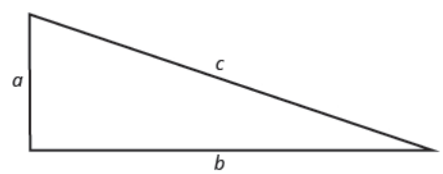
Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	 	
Step 2. Identify what you are looking for.	We are looking for the length and width.	
Step 3. Name what you are looking for. The length is two feet more than width.	Let W = the width of the garden. $W + 2$ = the length of the garden	
Step 4. Translate into an equation. Restate the important information in a sentence.	The area of the rectangular garden is 15 square feet.	
Use the formula for the area of a rectangle.	$A = L \cdot W$	
Substitute in the variables.	$15 = (W + 2)W$	
Step 5. Solve the equation. Distribute first.	$15 = W^2 + 2W$	
Get zero on one side.	$0 = W^2 + 2W - 15$	
Factor the trinomial.	$0 = (W + 5)(W - 3)$	
Use the Zero Product Property.	$0 = W + 5$	$0 = W - 3$
Solve each equation.	$-5 = W$	$3 = W$

Since *W* is the width of the garden, it does not make sense for it to be negative. We eliminate that value for *W*.	~~$-5 = W$~~ $W = 3$	$3 = W$ Width is 3 feet.
Find the value of the length.	$W + 2 = \text{length}$	
	$3 + 2$	
	5	Length is 5 feet.
Step 6. Check the answer. Does the answer make sense?		
 W 3 $W + 2$ $3 + 2$ 5 $A = L \cdot W$ $A = 3 \cdot 5$ $A = 15$		
	Yes, this makes sense.	
Step 7. Answer the question.	The width of the garden is 3 feet and the length is 5 feet.	

> **TRY IT : :** 7.161
>
> A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

> **TRY IT : :** 7.162
>
> A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

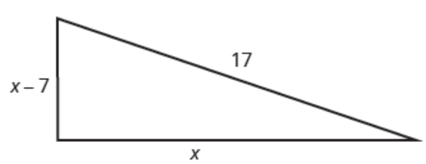
In an earlier chapter, we used the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$. It gave the relation between the legs and the hypotenuse of a right triangle.



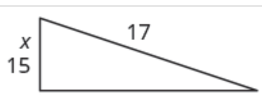
We will use this formula to in the next example.

EXAMPLE 7.82

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle, as shown below. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the deck.



Solution

Step 1. Read the problem.			
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the deck.		
Step 3. Name what you are looking for. One side is 7 less than the other.	Let x = length of a side of the deck $x - 7$ = length of other side		
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$		
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$		
Step 5. Solve the equation.	$x^2 + x^2 - 14x + 49 = 289$		
Simplify.	$2x^2 - 14x + 49 = 289$		
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$		
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$		
Factor the trinomial.	$2(x-15)(x+8) = 0$		
Use the Zero Product Property.	$2 \neq 0$	$x - 15 = 0$	$x + 8 = 0$
Solve.	$2 \neq 0$	$x = 15$	$x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0$	$x = 15$	~~$x = -8$~~
Find the length of the other side.			
If the length of one side is	$x = 15$		
then the length of the other side is	$x - 7$		
	$15 - 7$		
	8 is the length of the other side.		
Step 6. Check the answer. Do these numbers make sense?			
 $15 - 7$ 8	$a^2 + b^2 = c^2$ $15^2 + 8^2 \stackrel{?}{=} 17^2$ $225 + 64 \stackrel{?}{=} 289$ $289 = 289$ ✓		
Step 7. Answer the question.	The sides of the deck are 8, 15, and 17 feet.		

TRY IT : : 7.163

A boat's sail is a right triangle. The length of one side of the sail is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the sail.

TRY IT : : 7.164

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of one of the other legs. Find the lengths of the hypotenuse and the other leg.

7.6 EXERCISES

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

315. $(x-3)(x+7)=0$

316. $(y-11)(y+1)=0$

317. $(3a-10)(2a-7)=0$

318. $(5b+1)(6b+1)=0$

319. $6m(12m-5)=0$

320. $2x(6x-3)=0$

321. $(y-3)^2=0$

322. $(b+10)^2=0$

323. $(2x-1)^2=0$

324. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

325. $x^2+7x+12=0$

326. $y^2-8y+15=0$

327. $5a^2-26a=24$

328. $4b^2+7b=-3$

329. $4m^2=17m-15$

330. $n^2=5-6n \;\; n^2=5n-6$

331. $7a^2+14a=7a$

332. $12b^2-15b=-9b$

333. $49m^2=144$

334. $625=x^2$

335. $(y-3)(y+2)=4y$

336. $(p-5)(p+3)=-7$

337. $(2x+1)(x-3)=-4x$

338. $(x+6)(x-3)=-8$

339. $16p^3=24p^2+9p$

340. $m^3-2m^2=-m$

341. $20x^2-60x=-45$

342. $3y^2-18y=-27$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

343. The product of two consecutive integers is 56. Find the integers.

344. The product of two consecutive integers is 42. Find the integers.

345. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

346. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

347. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

348. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Mixed Practice

In the following exercises, solve.

349. $(x+8)(x-3)=0$

350. $(3y-5)(y+7)=0$

351. $p^2+12p+11=0$

352. $q^2 - 12q - 13 = 0$

353. $m^2 = 6m + 16$

354. $4n^2 + 19n = 5$

355. $a^3 - a^2 - 42a = 0$

356. $4b^2 - 60b + 224 = 0$

357. The product of two consecutive integers is 110. Find the integers.

358. The length of one leg of a right triangle is three more than the other leg. If the hypotenuse is 15, find the lengths of the two legs.

Everyday Math

359. Area of a patio If each side of a square patio is increased by 4 feet, the area of the patio would be 196 square feet. Solve the equation $(s + 4)^2 = 196$ for *s* to find the length of a side of the patio.

360. Watermelon drop A watermelon is dropped from the tenth story of a building. Solve the equation $-16t^2 + 144 = 0$ for t to find the number of seconds it takes the watermelon to reach the ground.

Writing Exercises

361. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

362. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

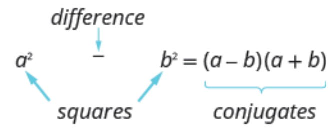
ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 7 REVIEW

KEY TERMS

difference of squares pattern If *a* and *b* are real numbers,

$$a^2 - b^2 = (a - b)(a + b)$$



factoring Factoring is splitting a product into factors; in other words, it is the reverse process of multiplying.

greatest common factor The greatest common factor is the largest expression that is a factor of two or more expressions is the greatest common factor (GCF).

perfect square trinomials pattern If *a* and *b* are real numbers,

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad a^2 - 2ab + b^2 = (a - b)^2$$

prime polynomials Polynomials that cannot be factored are prime polynomials.

quadratic equations are equations in which the variable is squared.

sum and difference of cubes pattern

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

Zero Product Property The Zero Product Property states that, if the product of two quantities is zero, at least one of the quantities is zero.

KEY CONCEPTS

7.1 Greatest Common Factor and Factor by Grouping

- **Finding the Greatest Common Factor (GCF):** To find the GCF of two expressions:
  Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.
  Step 3. Bring down the common factors that all expressions share.
  Step 4. Multiply the factors as in Example 7.2.
- **Factor the Greatest Common Factor from a Polynomial:** To factor a greatest common factor from a polynomial:
  Step 1. Find the GCF of all the terms of the polynomial.
  Step 2. Rewrite each term as a product using the GCF.
  Step 3. Use the 'reverse' Distributive Property to factor the expression.
  Step 4. Check by multiplying the factors as in Example 7.5.
- **Factor by Grouping:** To factor a polynomial with 4 four or more terms
  Step 1. Group terms with common factors.
  Step 2. Factor out the common factor in each group.
  Step 3. Factor the common factor from the expression.
  Step 4. Check by multiplying the factors as in Example 7.15.

7.2 Factor Quadratic Trinomials with Leading Coefficient 1

- **Factor trinomials of the form $x^2 + bx + c$**
  Step 1. Write the factors as two binomials with first terms *x*: $(x \quad)(x \quad)$.
  Step 2. Find two numbers *m* and *n* that
  Multiply to *c*, $m \cdot n = c$
  Add to *b*, $m + n = b$
  Step 3.

Use *m* and *n* as the last terms of the factors: $(x+m)(x+n)$.

Step 4. Check by multiplying the factors.

7.3 Factor Quadratic Trinomials with Leading Coefficient Other than 1

- **Factor Trinomials of the Form $ax^2 + bx + c$ using Trial and Error:** See Example 7.33.

  Step 1. Write the trinomial in descending order of degrees.

  Step 2. Find all the factor pairs of the first term.

  Step 3. Find all the factor pairs of the third term.

  Step 4. Test all the possible combinations of the factors until the correct product is found.

  Step 5. Check by multiplying.

- **Factor Trinomials of the Form $ax^2 + bx + c$ Using the "ac" Method:** See Example 7.38.

  Step 1. Factor any GCF.

  Step 2. Find the product ac.

  Step 3. Find two numbers *m* and *n* that:
  Multiply to ac $\quad m \cdot n = a \cdot c$
  Add to b $\quad m + n = b$

  Step 4. Split the middle term using *m* and *n*:

  $$ax^2 + bx + c$$
  $$ax^2 + \underbrace{mx + nx}_{bx} + c$$

  Step 5. Factor by grouping.

  Step 6. Check by multiplying the factors.

- **Choose a strategy to factor polynomials completely (updated):**

  Step 1. Is there a greatest common factor? Factor it.

  Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
  If it is a binomial, right now we have no method to factor it.
  If it is a trinomial of the form $x^2 + bx + c$
  Undo FOIL $(x \quad)(x \quad)$.
  If it is a trinomial of the form $ax^2 + bx + c$
  Use Trial and Error or the "ac" method.
  If it has more than three terms
  Use the grouping method.

  Step 3. Check by multiplying the factors.

7.4 Factor Special Products

- **Factor perfect square trinomials** See Example 7.42.

Step 1. Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$
Step 2. Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3. Check by multiplying.		

- **Factor differences of squares** See Example 7.47.

Step 1. Does the binomial fit he pattern? $a^2 - b^2$
Is this a diffe ence? $\_\_\_\_ - \_\_\_\_$
Are the fir t and last terms perfect squares?

Step 2. Write them as squares. $(a)^2 - (b)^2$

Step 3. Write the product of conjugates. $(a - b)(a + b)$

Step 4. Check by multiplying.

- **Factor sum and difference of cubes** To factor the sum or difference of cubes: See **Example 7.54**.

  Step 1. Does the binomial fit the sum or difference of cubes pattern? Is it a sum or difference? Are the first and last terms perfect cubes?

  Step 2. Write them as cubes.

  Step 3. Use either the sum or difference of cubes pattern.

  Step 4. Simplify inside the parentheses

  Step 5. Check by multiplying the factors.

7.5 General Strategy for Factoring Polynomials

- **General Strategy for Factoring Polynomials See Figure 7.4.**
- **How to Factor Polynomials**

  Step 1. Is there a greatest common factor? Factor it out.

  Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?

  - If it is a binomial:
    Is it a sum?
    - Of squares? Sums of squares do not factor.
    - Of cubes? Use the sum of cubes pattern.

    Is it a difference?
    - Of squares? Factor as the product of conjugates.
    - Of cubes? Use the difference of cubes pattern.
  - If it is a trinomial:
    Is it of the form $x^2 + bx + c$? Undo FOIL.

    Is it of the form $ax^2 + bx + c$?
    - If ‘a’ and ‘c’ are squares, check if it fits the trinomial square pattern.
    - Use the trial and error or ‘ac’ method.
  - If it has more than three terms:
    Use the grouping method.

  Step 3. Check. Is it factored completely? Do the factors multiply back to the original polynomial?

7.6 Quadratic Equations

- **Zero Product Property** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both. See **Example 7.69**.
- **Solve a quadratic equation by factoring** To solve a quadratic equation by factoring: See **Example 7.73**.

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.

  Step 2. Factor the quadratic expression.

  Step 3. Use the Zero Product Property.

  Step 4. Solve the linear equations.

  Step 5. Check.
- **Use a problem solving strategy to solve word problems** See **Example 7.80**.

  Step 1. **Read** the problem. Make sure all the words and ideas are understood.

  Step 2. **Identify** what we are looking for.

  Step 3.

Name what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

7.1 7.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

363. $42, 60$

364. $450, 420$

365. $90, 150, 105$

366. $60, 294, 630$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

367. $24x - 42$

368. $35y + 84$

369. $15m^4 + 6m^2 n$

370. $24pt^4 + 16t^7$

Factor by Grouping

In the following exercises, factor by grouping.

371. $ax - ay + bx - by$

372. $x^2 y - xy^2 + 2x - 2y$

373. $x^2 + 7x - 3x - 21$

374. $4x^2 - 16x + 3x - 12$

375. $m^3 + m^2 + m + 1$

376. $5x - 5y - y + x$

7.2 7.2 Factor Trinomials of the form $x^2 + bx + c$

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

377. $u^2 + 17u + 72$

378. $a^2 + 14a + 33$

379. $k^2 - 16k + 60$

380. $r^2 - 11r + 28$

381. $y^2 + 6y - 7$

382. $m^2 + 3m - 54$

383. $s^2 - 2s - 8$

384. $x^2 - 3x - 10$

Factor Trinomials of the Form $x^2 + bxy + cy^2$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

385. $x^2 + 12xy + 35y^2$

386. $u^2 + 14uv + 48v^2$

387. $a^2 + 4ab - 21b^2$

388. $p^2 - 5pq - 36q^2$

7.3 7.3 Factoring Trinomials of the form $ax^2 + bx + c$

Recognize a Preliminary Strategy to Factor Polynomials Completely

In the following exercises, identify the best method to use to factor each polynomial.

389. $y^2 - 17y + 42$

390. $12r^2 + 32r + 5$

391. $8a^3 + 72a$

392. $4m - mn - 3n + 12$

Factor Trinomials of the Form $ax^2 + bx + c$ with a GCF

In the following exercises, factor completely.

393. $6x^2 + 42x + 60$

394. $8a^2 + 32a + 24$

395. $3n^4 - 12n^3 - 96n^2$

396. $5y^4 + 25y^2 - 70y$

Factor Trinomials Using the "ac" Method

In the following exercises, factor.

397. $2x^2 + 9x + 4$

398. $3y^2 + 17y + 10$

399. $18a^2 - 9a + 1$

400. $8u^2 - 14u + 3$

401. $15p^2 + 2p - 8$

402. $15x^2 + 6x - 2$

403. $40s^2 - s - 6$

404. $20n^2 - 7n - 3$

Factor Trinomials with a GCF Using the "ac" Method

In the following exercises, factor.

405. $3x^2 + 3x - 36$

406. $4x^2 + 4x - 8$

407. $60y^2 - 85y - 25$

408. $18a^2 - 57a - 21$

7.4 7.4 Factoring Special Products

Factor Perfect Square Trinomials

In the following exercises, factor.

409. $25x^2 + 30x + 9$

410. $16y^2 + 72y + 81$

411. $36a^2 - 84ab + 49b^2$

412. $64r^2 - 176rs + 121s^2$

413. $40x^2 + 360x + 810$

414. $75u^2 + 180u + 108$

415. $2y^3 - 16y^2 + 32y$

416. $5k^3 - 70k^2 + 245k$

Factor Differences of Squares

In the following exercises, factor.

417. $81r^2 - 25$

418. $49a^2 - 144$

419. $169m^2 - n^2$

420. $64x^2 - y^2$

421. $25p^2 - 1$

422. $1 - 16s^2$

423. $9 - 121y^2$

424. $100k^2 - 81$

425. $20x^2 - 125$

426. $18y^2 - 98$

427. $49u^3 - 9u$

428. $169n^3 - n$

Factor Sums and Differences of Cubes

In the following exercises, factor.

429. $a^3 - 125$

430. $b^3 - 216$

431. $2m^3 + 54$

432. $81x^3 + 3$

7.5 7.5 General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

433. $24x^3 + 44x^2$

434. $24a^4 - 9a^3$

435. $16n^2 - 56mn + 49m^2$

436. $6a^2 - 25a - 9$

437. $5r^2 + 22r - 48$

438. $5u^4 - 45u^2$

439. $n^4 - 81$

440. $64j^2 + 225$

441. $5x^2 + 5x - 60$

442. $b^3 - 64$

443. $m^3 + 125$

444. $2b^2 - 2bc + 5cb - 5c^2$

7.6 7.6 Quadratic Equations

Use the Zero Product Property

In the following exercises, solve.

445. $(a - 3)(a + 7) = 0$

446. $(b - 3)(b + 10) = 0$

447. $3m(2m - 5)(m + 6) = 0$

448. $7n(3n + 8)(n - 5) = 0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

449. $x^2 + 9x + 20 = 0$

450. $y^2 - y - 72 = 0$

451. $2p^2 - 11p = 40$

452. $q^3 + 3q^2 + 2q = 0$

453. $144m^2 - 25 = 0$

454. $4n^2 = 36$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

455. The product of two consecutive numbers is 462. Find the numbers.

456. The area of a rectangular shaped patio 400 square feet. The length of the patio is 9 feet more than its width. Find the length and width.

PRACTICE TEST

In the following exercises, find the Greatest Common Factor in each expression.

457. $14y - 42$

458. $-6x^2 - 30x$

459. $80a^2 + 120a^3$

460. $5m(m - 1) + 3(m - 1)$

In the following exercises, factor completely.

461. $x^2 + 13x + 36$

462. $p^2 + pq - 12q^2$

463. $3a^3 - 6a^2 - 72a$

464. $s^2 - 25s + 84$

465. $5n^2 + 30n + 45$

466. $64y^2 - 49$

467. $xy - 8y + 7x - 56$

468. $40r^2 + 810$

469. $9s^2 - 12s + 4$

470. $n^2 + 12n + 36$

471. $100 - a^2$

472. $6x^2 - 11x - 10$

473. $3x^2 - 75y^2$

474. $c^3 - 1000d^3$

475. $ab - 3b - 2a + 6$

476. $6u^2 + 3u - 18$

477. $8m^2 + 22m + 5$

In the following exercises, solve.

478. $x^2 + 9x + 20 = 0$

479. $y^2 = y + 132$

480. $5a^2 + 26a = 24$

481. $9b^2 - 9 = 0$

482. $16 - m^2 = 0$

483. $4n^2 + 19 + 21 = 0$

484. $(x - 3)(x + 2) = 6$

485. The product of two consecutive integers is 156. Find the integers.

486. The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

8 RATIONAL EXPRESSIONS AND EQUATIONS

Figure 8.1 Rowing a boat downstream can be very relaxing, but it takes much more effort to row the boat upstream.

Chapter Outline

Introduction

Like rowing a boat, riding a bicycle is a situation in which going in one direction, downhill, is easy, but going in the opposite direction, uphill, can be more work. The trip to reach a destination may be quick, but the return trip whether upstream or uphill will take longer.

Rational equations are used to model situations like these. In this chapter, we will work with rational expressions, solve rational equations, and use them to solve problems in a variety of applications.

8.1 Simplify Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Evaluate rational expressions
- Simplify rational expressions
- Simplify rational expressions with opposite factors

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\frac{90y}{15y^2}$.

If you missed this problem, review **Example 6.66**.

2. Factor: $6x^2 - 7x + 2$.
If you missed this problem, review **Example 7.34**.
3. Factor: $n^3 + 8$.
If you missed this problem, review **Example 7.54**.

In Chapter 1, we reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers, and the denominator is not zero.

In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call these rational expressions.

Rational Expression

A **rational expression** is an expression of the form $\frac{p(x)}{q(x)}$, where *p* and *q* are polynomials and $q \neq 0$.

Remember, division by 0 is undefined.

Here are some examples of rational expressions:

$$-\frac{13}{42} \qquad \frac{7y}{8z} \qquad \frac{5x+2}{x^2-7} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{13}{42}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will perform same operations with rational expressions that we do with fractions. We will simplify, add, subtract, multiply, divide, and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

When we work with a numerical fraction, it is easy to avoid dividing by zero, because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

HOW TO : : DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

Step 1. Set the denominator equal to zero.

Step 2. Solve the equation in the set of reals, if possible.

EXAMPLE 8.1

Determine the values for which the rational expression is undefined:

ⓐ $\frac{9y}{x}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$

Solution

The expression will be undefined when the denominator is zero.

ⓐ

$$\frac{9y}{x}$$

Set the denominator equal to zero. Solve for the variable.

$$x = 0$$

$$\frac{9y}{x} \text{ is undefined or } x = 0.$$

ⓑ

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero. Solve for the variable.

$$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$$

$$\frac{4b-3}{2b+5} \text{ is undefined or } b = -\frac{5}{2}.$$

ⓒ

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero. Solve for the variable.

$$\begin{aligned} x^2+5x+6 &= 0 \\ (x+2)(x+3) &= 0 \\ x+2=0 \text{ or } x+3 &= 0 \\ x=-2 \text{ or } x &= -3 \end{aligned}$$

$$\frac{x+4}{x^2+5x+6} \text{ is undefined or } x=-2 \text{ or } x=-3.$$

Saying that the rational expression $\frac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$ is similar to writing the phrase "void where prohibited" in contest rules.

TRY IT : : 8.1 Determine the values for which the rational expression is undefined:

ⓐ $\frac{3y}{x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

TRY IT : : 8.2 Determine the values for which the rational expression is undefined:

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Evaluate Rational Expressions

To evaluate a rational expression, we substitute values of the variables into the expression and simplify, just as we have for many other expressions in this book.

EXAMPLE 8.2

Evaluate $\frac{2x+3}{3x-5}$ for each value:

ⓐ $x=0$ ⓑ $x=2$ ⓒ $x=-3$

Solution

ⓐ

	$\frac{2x+3}{3x-5}$
Substitute 0 for *x*.	$\frac{2(0)+3}{3(0)-5}$
Simplify.	$-\frac{3}{5}$

ⓑ

	$\frac{2x+3}{3x-5}$
Substitute 2 for *x*.	$\frac{2(2)+3}{3(2)-5}$
Simplify.	$\frac{4+3}{6-5}$
	$\frac{7}{1}$
	7

ⓒ

	$\frac{2x+3}{3x-5}$
Substitute −3 for *x*.	$\frac{2(-3)+3}{3(-3)-5}$
Simplify.	$\frac{-6+3}{-9-5}$
	$\frac{-3}{-14}$
	$\frac{3}{14}$

TRY IT : : 8.3

Evaluate $\frac{y+1}{2y-3}$ for each value:

ⓐ $y=1$ ⓑ $y=-3$ ⓒ $y=0$

TRY IT : : 8.4

Evaluate $\frac{5x-1}{2x+1}$ for each value:

ⓐ $x=1$ ⓑ $x=-1$ ⓒ $x=0$

EXAMPLE 8.3

Evaluate $\frac{x^2+8x+7}{x^2-4}$ for each value:

ⓐ $x=0$ ⓑ $x=2$ ⓒ $x=-1$

Solution

ⓐ

	$\dfrac{x^2+8x+7}{x^2-4}$
Substitute 0 for x.	$\dfrac{(0)^2+8(0)+7}{(0)^2-4}$
Simplify.	$\dfrac{7}{-4}$
	$-\dfrac{7}{4}$

ⓑ

	$\dfrac{x^2+8x+7}{x^2-4}$
Substitute 2 for x.	$\dfrac{(2)^2+8(2)+7}{(2)^2-4}$
Simplify.	$\dfrac{4+16+7}{4-4}$
	$\dfrac{27}{0}$
This rational expression is undefined for $x=2$.	

ⓒ

	$\dfrac{x^2+8x+7}{x^2-4}$
Substitute −1 for x.	$\dfrac{(-1)^2+8(-1)+7}{(-1)^2-4}$
Simplify.	$\dfrac{1-8+7}{1-4}$
	$\dfrac{-7+7}{-3}$
	$\dfrac{0}{-3}$
	0

TRY IT : : 8.5

Evaluate $\dfrac{x^2+1}{x^2-3x+2}$ for each value:

ⓐ $x=0$ ⓑ $x=-1$ ⓒ $x=3$

TRY IT : : 8.6

Evaluate $\dfrac{x^2+x-6}{x^2-9}$ for each value:

ⓐ $x=0$ ⓑ $x=-2$ ⓒ $x=1$

Remember that a fraction is simplified when it has no common factors, other than 1, in its numerator and denominator. When we evaluate a rational expression, we make sure to simplify the resulting fraction.

EXAMPLE 8.4

Evaluate $\dfrac{a^2+2ab+b^2}{3ab^3}$ for each value:

ⓐ $a=1,\ b=2$ ⓑ $a=-2,\ b=-1$ ⓒ $a=\frac{1}{3},\ b=0$

Solution

ⓐ

	$\dfrac{a^2+2ab+b^2}{3ab^2}$ when $a=1,\ b=2$.
Substitute 1 for *a* and 2 for *b*.	$\dfrac{(1)^2+2(1)(2)+(2)^2}{3(1)(2)^2}$
Simplify.	$\dfrac{1+4+4}{3(4)}$
	$\dfrac{9}{12}$
	$\dfrac{3}{4}$

ⓑ

	$\dfrac{a^2+2ab+b^2}{3ab^2}$ when $a=-2,\ b=-1$.
Substitute −2 for *a* and −1 for *b*.	$\dfrac{(-2)^2+2(-2)(-1)+(-1)^2}{3(-2)(-1)^2}$
Simplify.	$\dfrac{4+4+1}{-6}$
	$-\dfrac{9}{6}$
	$-\dfrac{3}{2}$

ⓒ

	$\dfrac{a^2+2ab+b^2}{3ab^2}$ when $a=\frac{1}{3},\ b=0$.
Substitute $\frac{1}{3}$ for *a* and 0 for *b*.	$\dfrac{\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)(0)+(0)^2}{3\left(\frac{1}{3}\right)(0)^2}$

Simplify.	$\dfrac{\frac{1}{9}+0+0}{0}$
	$\dfrac{\frac{1}{9}}{0}$
	The expression is undefined.

TRY IT : : 8.7

Evaluate $\dfrac{2a^3 b}{a^2+2ab+b^2}$ for each value:

ⓐ $a=-1,\ b=2$ ⓑ $a=0,\ b=-1$ ⓒ $a=1,\ b=\frac{1}{2}$

TRY IT : : 8.8

Evaluate $\dfrac{a^2-b^2}{8ab^3}$ for each value:

ⓐ $a=1,\ b=-1$ ⓑ $a=\frac{1}{2},\ b=-1$ ⓒ $a=-2,\ b=1$

Simplify Rational Expressions

Just like a fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator, a rational expression is *simplified* if it has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example:

- $\frac{2}{3}$ is simplified because there are no common factors of 2 and 3.
- $\frac{2x}{3x}$ is not simplified because *x* is a common factor of 2*x* and 3*x*.

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If *a*, *b*, and *c* are numbers where $b \neq 0,\ c \neq 0$, then $\frac{a}{b}=\frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c}=\frac{a}{b}$.

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0,\ c \neq 0$ clearly stated. Every time we write a rational expression, we should make a similar statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Let's start by reviewing how we simplify numerical fractions.

EXAMPLE 8.5

Simplify: $-\frac{36}{63}$.

Solution

	$-\frac{36}{63}$
Rewrite the numerator and denominator showing the common factors.	$-\frac{4 \cdot 9}{7 \cdot 9}$
Simplify using the Equivalent Fractions Property.	$-\frac{4}{7}$

Notice that the fraction $-\frac{4}{7}$ is simplified because there are no more common factors.

TRY IT : : 8.9 Simplify: $-\frac{45}{81}$.

TRY IT : : 8.10 Simplify: $-\frac{42}{54}$.

Throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

EXAMPLE 8.6

Simplify: $\frac{3xy}{18x^2y^2}$.

Solution

	$\frac{3xy}{18x^2y^2}$
Rewrite the numerator and denominator showing the common factors.	$\frac{1 \cdot 3xy}{6xy \cdot 3xy}$
Simplify using the Equivalent Fractions Property.	$\frac{1}{6xy}$

Did you notice that these are the same steps we took when we divided monomials in **Polynomials**?

TRY IT : : 8.11 Simplify: $\frac{4x^2y}{12xy^2}$.

TRY IT : : 8.12 Simplify: $\frac{16x^2y}{2xy^2}$.

To simplify rational expressions we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\dfrac{2\cdot\cancel{3}\cdot\cancel{7}}{\cancel{3}\cdot 5\cdot\cancel{7}}$	$\dfrac{3x(x-9)}{5(x-9)}$	$\dfrac{x+5}{x}$
$\dfrac{2}{5}$	$\dfrac{3x}{5}$	**NO COMMON FACTORS**
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Note that removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

EXAMPLE 8.7 HOW TO SIMPLIFY RATIONAL BINOMIALS

Simplify: $\dfrac{2x+8}{5x+20}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $2x+8$ and $5x-20$.	$\dfrac{2x+8}{5x+20}$ $\dfrac{2(x+4)}{5(x+4)}$
Step 2. Simplify by dividing out common factors.	Divide out the common factors.	$\dfrac{2\cancel{(x+4)}}{5\cancel{(x+4)}}$ $\dfrac{2}{5}$

TRY IT : : 8.13 Simplify: $\dfrac{3x-6}{2x-4}$.

TRY IT : : 8.14 Simplify: $\dfrac{7y+35}{5y+25}$.

We now summarize the steps you should follow to simplify rational expressions.

HOW TO : : SIMPLIFY A RATIONAL EXPRESSION.

Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way it is easy to check that we have removed all the common factors!

We'll use the methods we covered in **Factoring** to factor the polynomials in the numerators and denominators in the following examples.

EXAMPLE 8.8

Simplify: $\dfrac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\dfrac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator.	$\dfrac{(x+2)(x+3)}{(x+2)(x+6)}$
Remove the common factor $x+2$ from the numerator and the denominator.	$\dfrac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$
	$\dfrac{x+3}{x+6}$

Can you tell which values of x must be excluded in this example?

TRY IT : : 8.15 Simplify: $\dfrac{x^2-x-2}{x^2-3x+2}$.

TRY IT : : 8.16 Simplify: $\dfrac{x^2-3x-10}{x^2+x-2}$.

EXAMPLE 8.9

Simplify: $\dfrac{y^2+y-42}{y^2-36}$.

Solution

	$\dfrac{y^2+y-42}{y^2-36}$
Factor the numerator and denominator.	$\dfrac{(y+7)(y-6)}{(y+6)(y-6)}$
Remove the common factor $y-6$ from the numerator and the denominator.	$\dfrac{(y+7)\cancel{(y-6)}}{(y+6)\cancel{(y-6)}}$
	$\dfrac{y+7}{y+6}$

TRY IT : : 8.17 Simplify: $\dfrac{x^2+x-6}{x^2-4}$.

TRY IT : : 8.18 Simplify: $\dfrac{x^2+8x+7}{x^2-49}$.

EXAMPLE 8.10

Simplify: $\dfrac{p^3-2p^2+2p-4}{p^2-7p+10}$.

Solution

$$\frac{p^3 - 2p^2 + 2p - 4}{p^2 - 7p + 10}$$

Factor the numerator and denominator, using grouping to factor the numerator.

$$\frac{p^2(p-2) + 2(p-2)}{(p-5)(p-2)}$$

$$\frac{(p^2+2)(p-2)}{(p-5)(p-2)}$$

Remove the common factor of $p - 2$ from the numerator and the denominator.

$$\frac{(p^2+2)\cancel{(p-2)}}{(p-5)\cancel{(p-2)}}$$

$$\frac{p^2+2}{p-5}$$

TRY IT : : 8.19 Simplify: $\dfrac{y^3 - 3y^2 + y - 3}{y^2 - y - 6}$.

TRY IT : : 8.20 Simplify: $\dfrac{p^3 - p^2 + 2p - 2}{p^2 + 4p - 5}$.

EXAMPLE 8.11

Simplify: $\dfrac{2n^2 - 14n}{4n^2 - 16n - 48}$.

Solution

$$\frac{2n^2 - 14n}{4n^2 - 16n - 48}$$

Factor the numerator and denominator, fir t factoring out the GCF.

$$\frac{2n(n-7)}{4(n^2 - 4n - 12)}$$

$$\frac{2n(n-7)}{4(n-6)(n+2)}$$

Remove the common factor, 2.

$$\frac{\cancel{2}n(n-7)}{\cancel{2}\cdot 2(n-6)(n+2)}$$

$$\frac{n(n-7)}{2(n-6)(n+2)}$$

TRY IT : : 8.21 Simplify: $\dfrac{2n^2 - 10n}{4n^2 - 16n - 20}$.

TRY IT : : 8.22 Simplify: $\dfrac{4x^2 - 16x}{8x^2 - 16x - 64}$.

EXAMPLE 8.12

Simplify: $\frac{3b^2 - 12b + 12}{6b^2 - 24}$.

Solution

	$\frac{3b^2 - 12b + 12}{6b^2 - 24}$
Factor the numerator and denominator, fir t factoring out the GCF.	$\frac{3(b^2 - 4b + 4)}{6(b^2 - 4)}$
	$\frac{3(b-2)(b-2)}{6(b+2)(b-2)}$
Remove the common factors of $b - 2$ and 3.	$\frac{\cancel{3}(b-2)\cancel{(b-2)}}{\cancel{3} \cdot 2(b+2)\cancel{(b-2)}}$
	$\frac{b-2}{2(b+2)}$

TRY IT : : 8.23 Simplify: $\frac{2x^2 - 12x + 18}{3x^2 - 27}$.

TRY IT : : 8.24 Simplify: $\frac{5y^2 - 30y + 25}{2y^2 - 50}$.

EXAMPLE 8.13

Simplify: $\frac{m^3 + 8}{m^2 - 4}$.

Solution

	$\frac{m^3 + 8}{m^2 - 4}$
Factor the numerator and denominator, using the formulas for sum of cubes and diffe ence of squares.	$\frac{(m+2)(m^2 - 2m + 4)}{(m+2)(m-2)}$
Remove the common factor of $m + 2$.	$\frac{\cancel{(m+2)}(m^2 - 2m + 4)}{\cancel{(m+2)}(m-2)}$
	$\frac{m^2 - 2m + 4}{m - 2}$

TRY IT : : 8.25 Simplify: $\frac{p^3 - 64}{p^2 - 16}$.

TRY IT : : 8.26 Simplify: $\frac{x^3 + 8}{x^2 - 4}$.

Simplify Rational Expressions with Opposite Factors

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. Let’s start with a numerical fraction, say $\frac{7}{-7}$. We know this fraction simplifies to -1. We also recognize that the numerator and denominator are opposites.

In **Foundations**, we introduced opposite notation: the opposite of a is $-a$. We remember, too, that $-a = -1 \cdot a$.

We simplify the fraction $\frac{a}{-a}$, whose numerator and denominator are opposites, in this way:

	$\frac{a}{-a}$
We could rewrite this.	$\frac{1 \cdot a}{-1 \cdot a}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

So, in the same way, we can simplify the fraction $\frac{x-3}{-(x-3)}$:

We could rewrite this.	$\frac{1 \cdot (x-3)}{-1 \cdot (x-3)}$
Remove the common factors.	$\frac{1}{-1}$
Simplify.	-1

But the opposite of $x-3$ could be written differently:

	$-(x-3)$
Distribute.	$-x+3$
Rewrite.	$3-x$

This means the fraction $\frac{x-3}{3-x}$ simplifies to -1.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a-b$ is $b-a$.

$$\frac{a-b}{b-a} = -1 \qquad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators.

EXAMPLE 8.14

Simplify: $\frac{x-8}{8-x}$.

Solution

	$\frac{x-8}{8-x}$
Recognize that $x-8$ and $8-x$ are opposites.	-1

TRY IT : : 8.27 Simplify: $\frac{y-2}{2-y}$.

TRY IT : : 8.28 Simplify: $\frac{n-9}{9-n}$.

Remember, the first step in simplifying a rational expression is to factor the numerator and denominator completely.

EXAMPLE 8.15

Simplify: $\dfrac{14-2x}{x^2-49}$.

Solution

	$\dfrac{14-2x}{x^2-49}$
Factor the numerator and denominator.	$\dfrac{2(7-x)}{(x+7)(x-7)}$
Recognize that $7-x$ and $x-7$ are opposites .	$\dfrac{2\cancel{(7-x)}}{(x+7)\cancel{(x-7)}}(-1)$
Simplify.	$-\dfrac{2}{x+7}$

TRY IT : : 8.29 Simplify: $\dfrac{10-2y}{y^2-25}$.

TRY IT : : 8.30 Simplify: $\dfrac{3y-27}{81-y^2}$.

EXAMPLE 8.16

Simplify: $\dfrac{x^2-4x-32}{64-x^2}$.

Solution

	$\dfrac{x^2-4x-32}{64-x^2}$
Factor the numerator and denominator.	$\dfrac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\dfrac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$-\dfrac{x+4}{x+8}$

TRY IT : : 8.31 Simplify: $\dfrac{x^2-4x-5}{25-x^2}$.

TRY IT : : 8.32 Simplify: $\dfrac{x^2+x-2}{1-x^2}$.

8.1 EXERCISES

Practice Makes Perfect

In the following exercises, determine the values for which the rational expression is undefined.

1.
ⓐ $\frac{2x}{z}$
ⓑ $\frac{4p-1}{6p-5}$
ⓒ $\frac{n-3}{n^2+2n-8}$

2.
ⓐ $\frac{10m}{11n}$
ⓑ $\frac{6y+13}{4y-9}$
ⓒ $\frac{b-8}{b^2-36}$

3.
ⓐ $\frac{4x^2y}{3y}$
ⓑ $\frac{3x-2}{2x+1}$
ⓒ $\frac{u-1}{u^2-3u-28}$

4.
ⓐ $\frac{5pq^2}{9q}$
ⓑ $\frac{7a-4}{3a+5}$
ⓒ $\frac{1}{x^2-4}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expression for the given values.

5. $\frac{2x}{x-1}$
ⓐ $x=0$
ⓑ $x=2$
ⓒ $x=-1$

6. $\frac{4y-1}{5y-3}$
ⓐ $y=0$
ⓑ $y=2$
ⓒ $y=-1$

7. $\frac{2p+3}{p^2+1}$
ⓐ $p=0$
ⓑ $p=1$
ⓒ $p=-2$

8. $\frac{x+3}{2-3x}$
ⓐ $x=0$
ⓑ $x=1$
ⓒ $x=-2$

9. $\frac{y^2+5y+6}{y^2-1}$
ⓐ $y=0$
ⓑ $y=2$
ⓒ $y=-2$

10. $\frac{z^2+3z-10}{z^2-1}$
ⓐ $z=0$
ⓑ $z=2$
ⓒ $z=-2$

11. $\frac{a^2-4}{a^2+5a+4}$
ⓐ $a=0$
ⓑ $a=1$
ⓒ $a=-2$

12. $\frac{b^2+2}{b^2-3b-4}$
ⓐ $b=0$
ⓑ $b=2$
ⓒ $b=-2$

13. $\frac{x^2+3xy+2y^2}{2x^3y}$
ⓐ $x=1,\ y=-1$
ⓑ $x=2,\ y=1$
ⓒ $x=-1,\ y=-2$

14. $\dfrac{c^2 + cd - 2d^2}{cd^3}$

ⓐ $c = 2,\ d = -1$

ⓑ $c = 1,\ d = -1$

ⓒ $c = -1,\ d = 2$

15. $\dfrac{m^2 - 4n^2}{5mn^3}$

ⓐ $m = 2,\ n = 1$

ⓑ $m = -1,\ n = -1$

ⓒ $m = 3,\ n = 2$

16. $\dfrac{2s^2 t}{s^2 - 9t^2}$

ⓐ $s = 4,\ t = 1$

ⓑ $s = -1,\ t = -1$

ⓒ $s = 0,\ t = 2$

Simplify Rational Expressions

In the following exercises, simplify.

17. $-\dfrac{4}{52}$

18. $-\dfrac{44}{55}$

19. $\dfrac{56}{63}$

20. $\dfrac{65}{104}$

21. $\dfrac{6ab^2}{12a^2 b}$

22. $\dfrac{15xy}{3x^3 y^3}$

23. $\dfrac{8m^3 n}{12mn^2}$

24. $\dfrac{36v^3 w^2}{27vw^3}$

25. $\dfrac{3a + 6}{4a + 8}$

26. $\dfrac{5b + 5}{6b + 6}$

27. $\dfrac{3c - 9}{5c - 15}$

28. $\dfrac{4d + 8}{9d + 18}$

29. $\dfrac{7m + 63}{5m + 45}$

30. $\dfrac{8n - 96}{3n - 36}$

31. $\dfrac{12p - 240}{5p - 100}$

32. $\dfrac{6q + 210}{5q + 175}$

33. $\dfrac{a^2 - a - 12}{a^2 - 8a + 16}$

34. $\dfrac{x^2 + 4x - 5}{x^2 - 2x + 1}$

35. $\dfrac{y^2 + 3y - 4}{y^2 - 6y + 5}$

36. $\dfrac{v^2 + 8v + 15}{v^2 - v - 12}$

37. $\dfrac{x^2 - 25}{x^2 + 2x - 15}$

38. $\dfrac{a^2 - 4}{a^2 + 6a - 16}$

39. $\dfrac{y^2 - 2y - 3}{y^2 - 9}$

40. $\dfrac{b^2 + 9b + 18}{b^2 - 36}$

41. $\dfrac{y^3 + y^2 + y + 1}{y^2 + 2y + 1}$

42. $\dfrac{p^3 + 3p^2 + 4p + 12}{p^2 + p - 6}$

43. $\dfrac{x^3 - 2x^2 - 25x + 50}{x^2 - 25}$

44. $\dfrac{q^3 + 3q^2 - 4q - 12}{q^2 - 4}$

45. $\dfrac{3a^2 + 15a}{6a^2 + 6a - 36}$

46. $\dfrac{8b^2 - 32b}{2b^2 - 6b - 80}$

47. $\dfrac{-5c^2 - 10c}{-10c^2 + 30c + 100}$

48. $\dfrac{4d^2 - 24d}{2d^2 - 4d - 48}$

49. $\dfrac{3m^2 + 30m + 75}{4m^2 - 100}$

50. $\dfrac{5n^2 + 30n + 45}{2n^2 - 18}$

51. $\dfrac{5r^2 + 30r - 35}{r^2 - 49}$

52. $\dfrac{3s^2 + 30s + 24}{3s^2 - 48}$

53. $\dfrac{t^3 - 27}{t^2 - 9}$

54. $\dfrac{v^3 - 1}{v^2 - 1}$

55. $\dfrac{w^3 + 216}{w^2 - 36}$

56. $\frac{v^3 + 125}{v^2 - 25}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify each rational expression.

57. $\frac{a-5}{5-a}$

58. $\frac{b-12}{12-b}$

59. $\frac{11-c}{c-11}$

60. $\frac{5-d}{d-5}$

61. $\frac{12-2x}{x^2-36}$

62. $\frac{20-5y}{y^2-16}$

63. $\frac{4v-32}{64-v^2}$

64. $\frac{7w-21}{9-w^2}$

65. $\frac{y^2-11y+24}{9-y^2}$

66. $\frac{z^2-9z+20}{16-z^2}$

67. $\frac{a^2-5z-36}{81-a^2}$

68. $\frac{b^2+b-42}{36-b^2}$

Everyday Math

69. Tax Rates For the tax year 2015, the amount of tax owed by a single person earning between $37,450 and $90,750, can be found by evaluating the formula $0.25x - 4206.25,$ where *x* is income. The average tax rate for this income can be found by evaluating the formula $\frac{0.25x - 4206.25}{x}.$ What would be the average tax rate for a single person earning $50,000?

70. Work The length of time it takes for two people for perform the same task if they work together can be found by evaluating the formula $\frac{xy}{x+y}.$ If Tom can paint the den in $x =$ 45 minutes and his brother Bobby can paint it in $y =$ 60 minutes, how many minutes will it take them if they work together?

Writing Exercises

71. Explain how you find the values of *x* for which the rational expression $\frac{x^2 - x - 20}{x^2 - 4}$ is undefined.

72. Explain all the steps you take to simplify the rational expression $\frac{p^2 + 4p - 21}{9 - p^2}.$

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
evaluate rational expressions.			
simplify rational expressions.			
simplify rational expressions with opposite factors.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math* *tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

8.2 Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Multiply rational expressions
- Divide rational expressions

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
   If you missed this problem, review Example 1.68.
2. Divide: $\frac{14}{15} \div \frac{6}{35}$.
   If you missed this problem, review Example 1.71.
3. Factor completely: $2x^2 - 98$.
   If you missed this problem, review Example 7.62.
4. Factor completely: $10n^3 + 10$.
   If you missed this problem, review Example 7.65.
5. Factor completely: $10p^2 - 25pq - 15q^2$.
   If you missed this problem, review Example 7.68.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, s are polynomials where $q \neq 0$ and $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

We'll do the first example with numerical fractions to remind us of how we multiplied fractions without variables.

EXAMPLE 8.17

Multiply: $\frac{10}{28} \cdot \frac{8}{15}$.

Solution

	$\frac{10}{28} \cdot \frac{8}{15}$
Multiply the numerators and denominators.	$\frac{10 \cdot 8}{28 \cdot 15}$
Look for common factors, and then remove them.	$\frac{2 \cdot \cancel{5} \cdot 2 \cdot \cancel{4}}{7 \cdot \cancel{4} \cdot 3 \cdot \cancel{5}}$
Simplify.	$\frac{4}{21}$

TRY IT : : 8.33 Mulitply: $\frac{6}{10} \cdot \frac{15}{12}$.

TRY IT : : 8.34 Mulitply: $\frac{20}{15} \cdot \frac{6}{8}$.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$ and $y \neq 0$.

EXAMPLE 8.18

Mulitply: $\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2 y}$.

Solution

	$\frac{2x}{3y^2} \cdot \frac{6xy^3}{x^2 y}$
Multiply.	$\frac{2x \cdot 6xy^3}{3y^2 \cdot x^2 y}$
Factor the numerator and denominator completely, and then remove common factors.	$\frac{2 \cdot \cancel{x} \cdot 2 \cdot \cancel{3} \cdot \cancel{x} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{y}}{\cancel{3} \cdot \cancel{y} \cdot \cancel{y} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{y}}$
Simplify.	4

TRY IT : : 8.35 Mulitply: $\frac{3pq}{q^2} \cdot \frac{5p^2 q}{6pq}$.

TRY IT : : 8.36 Mulitply: $\frac{6x^3 y}{7x^2} \cdot \frac{2xy^3}{x^2 y}$.

EXAMPLE 8.19 HOW TO MULTIPLY RATIONAL EXPRESSIONS

Mulitply: $\frac{2x}{x^2 + x + 12} \cdot \frac{x^2 - 9}{6x^2}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor $x^2 - 9$ and $x^2 + x + 12$.	$\frac{2x}{x^2 + x + 12} \cdot \frac{x^2 - 9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\cancel{x}\,\cancel{(x-3)}(x+3)}{\cancel{2} \cdot 3 \cdot \cancel{x} \cdot x\,\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

TRY IT : : 8.37 Mulitply: $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

TRY IT : : 8.38 Mulitply: $\frac{9x^2}{x^2+11x+30} \cdot \frac{x^2-36}{3x^2}$.

HOW TO : : MULTIPLY A RATIONAL EXPRESSION.

Step 1. Factor each numerator and denominator completely.

Step 2. Multiply the numerators and denominators.

Step 3. Simplify by dividing out common factors.

EXAMPLE 8.20

Multiply: $\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$.

Solution

	$\frac{n^2-7n}{n^2+2n+1} \cdot \frac{n+1}{2n}$
Factor each numerator and denominator.	$\frac{n(n-7)}{(n+1)(n+1)} \cdot \frac{n+1}{2n}$
Multiply the numerators and the denominators.	$\frac{n(n-7)(n+1)}{(n+1)(n+1)2n}$
Remove common factors.	$\frac{\cancel{n}(n-7)\cancel{(n+1)}}{(n+1)\cancel{(n+1)}2\cancel{n}}$
Simplify.	$\frac{n-7}{2(n+1)}$

TRY IT : : 8.39 Multiply: $\frac{x^2-25}{x^2-3x-10} \cdot \frac{x+2}{x}$.

TRY IT : : 8.40 Multiply: $\frac{x^2-4x}{x^2+5x+6} \cdot \frac{x+2}{x}$.

EXAMPLE 8.21

Multiply: $\frac{16-4x}{2x-12} \cdot \frac{x^2-5x-6}{x^2-16}$.

Solution

	$\frac{16-4x}{2x-12}\cdot\frac{x^2-5x-6}{x^2-16}$
Factor each numerator and denominator.	$\frac{4(4-x)}{2(x-6)}\cdot\frac{(x-6)(x+1)}{(x-4)(x+4)}$
Multiply the numerators and the denominators.	$\frac{4(4-x)(x-6)(x+1)}{2(x-6)(x-4)(x+4)}$
Remove common factors.	$(-1)\frac{\cancel{2}\cdot 2\cancel{(4-x)}\cancel{(x-6)}(x+1)}{\cancel{2}\cancel{(x-6)}\cancel{(x-4)}(x+4)}$
Simplify.	$-\frac{2(x+1)}{(x+4)}$

TRY IT : : 8.41 Multiply: $\frac{12x-6x^2}{x^2+8x}\cdot\frac{x^2+11x+24}{x^2-4}$.

TRY IT : : 8.42 Multiply: $\frac{9v-3v^2}{9v+36}\cdot\frac{v^2+7v+12}{v^2-9}$.

EXAMPLE 8.22

Multiply: $\frac{2x-6}{x^2-8x+15}\cdot\frac{x^2-25}{2x+10}$.

Solution

	$\frac{2x-6}{x^2-8x+15}\cdot\frac{x^2-25}{2x+10}$
Factor each numerator and denominator.	$\frac{2(x-3)}{(x-3)(x-5)}\cdot\frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$\frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$\frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	1

TRY IT : : 8.43 Multiply: $\frac{3a-21}{a^2-9a+14}\cdot\frac{a^2-4}{3a+6}$.

TRY IT : : 8.44 Multiply: $\frac{b^2-b}{b^2+9b-10}\cdot\frac{b^2-100}{b^2-10b}$.

Divide Rational Expressions

To divide rational expressions we multiply the first fraction by the reciprocal of the second, just like we did for numerical fractions.

Remember, the **reciprocal** of $\frac{a}{b}$ is $\frac{b}{a}$. To find the reciprocal we simply put the numerator in the denominator and the denominator in the numerator. We "flip" the fraction.

Division of Rational Expressions

If p, q, r, s are polynomials where $q \neq 0,\ r \neq 0,\ s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions multiply the first fraction by the reciprocal of the second.

EXAMPLE 8.23 HOW TO DIVIDE RATIONAL EXPRESSIONS

Divide: $\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{x+9}{6-x} \div \frac{x^2-81}{x-6}$ $\frac{x+9}{6-x} \cdot \frac{x-6}{x^2-81}$
Step 2. Factor the numerators and denominators completely.	Factor $x^2 - 81$.	$\frac{x+9}{6-x} \cdot \frac{x-6}{(x-9)(x+9)}$
Step 3. Multiply the numerators and denominators.		$\frac{(x+9)(x-6)}{(6-x)(x-9)(x+9)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors. Remember opposites divide to −1.	$(-1)\frac{\cancel{(x+9)}\cancel{(x-6)}}{\cancel{(6-x)}(x-9)\cancel{(x+9)}}$ $-\frac{1}{(x-9)}$

TRY IT : : 8.45

Divide: $\frac{c+3}{5-c} \div \frac{c^2-9}{c-5}$.

TRY IT : : 8.46

Divide: $\frac{2-d}{d-4} \div \frac{4-d^2}{4-d}$.

HOW TO : : DIVIDE RATIONAL EXPRESSIONS.

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

EXAMPLE 8.24

Divide: $\frac{3n^2}{n^2 - 4n} \div \frac{9n^2 - 45n}{n^2 - 7n + 10}$.

Solution

	$\frac{3n^2}{n^2 - 4n} \div \frac{9n^2 - 45n}{n^2 - 7n + 10}$
Rewrite the division as the product of the first rational expression and the reciprocal of the second.	$\frac{3n^2}{n^2 - 4n} \cdot \frac{n^2 - 7n + 10}{9n^2 - 45n}$
Factor the numerators and denominators and then multiply.	$\frac{3 \cdot n \cdot n \cdot (n-5)(n-2)}{n(n-4) \cdot 3 \cdot 3 \cdot n \cdot (n-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{3} \cdot \cancel{n} \cdot \cancel{n}\cancel{(n-5)}(n-2)}{\cancel{n}(n-4)\cancel{3} \cdot 3 \cdot \cancel{n}\cancel{(n-5)}}$
	$\frac{n-2}{3(n-4)}$

TRY IT : : 8.47 Divide: $\frac{2m^2}{m^2 - 8m} \div \frac{8m^2 + 24m}{m^2 + m - 6}$.

TRY IT : : 8.48 Divide: $\frac{15n^2}{3n^2 + 33n} \div \frac{5n - 5}{n^2 + 9n - 22}$.

Remember, first rewrite the division as multiplication of the first expression by the reciprocal of the second. Then factor everything and look for common factors.

EXAMPLE 8.25

Divide: $\frac{2x^2 + 5x - 12}{x^2 - 16} \div \frac{2x^2 - 13x + 15}{x^2 - 8x + 16}$.

Solution

	$\frac{2x^2 + 5x - 12}{x^2 - 16} \div \frac{2x^2 - 13x + 15}{x^2 - 8x + 16}$
Rewrite the division as multiplication of the fir t expression by the reciprocal of the second.	$\frac{2x^2 + 5x - 12}{x^2 - 16} \cdot \frac{x^2 - 8x + 16}{2x^2 - 13x + 15}$
Factor the numerators and denominators and then multiply.	$\frac{(2x-3)(x+4)(x-4)(x-4)}{(x-4)(x+4)(2x-3)(x-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{(2x-3)}\cancel{(x+4)}\cancel{(x-4)}(x-4)}{\cancel{(x-4)}\cancel{(x+4)}\cancel{(2x-3)}(x-5)}$
Simplify.	$\frac{x-4}{x-5}$

TRY IT : : 8.49 Divide: $\frac{3a^2 - 8a - 3}{a^2 - 25} \div \frac{3a^2 - 14a - 5}{a^2 + 10a + 25}$.

TRY IT : : 8.50 Divide: $\frac{4b^2+7b-2}{1-b^2} \div \frac{4b^2+15b-4}{b^2-2b+1}$.

EXAMPLE 8.26

Divide: $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

$$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$$

Rewrite the division as a multiplication of the fir t expression times the reciprocal of the second.

$$\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$$

Factor the numerators and denominators and then multiply.

$$\frac{(p+q)(p^2-pq+q^2)6}{2(p^2+pq+q^2)(p-q)(p+q)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{6}^{3}}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$$

Simplify.

$$\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$$

TRY IT : : 8.51 Divide: $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

TRY IT : : 8.52 Divide: $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3-1}$.

Before doing the next example, let’s look at how we divide a fraction by a whole number. When we divide $\frac{3}{5} \div 4$, we first write 4 as a fraction so that we can find its reciprocal.

$$\frac{3}{5} \div 4$$

$$\frac{3}{5} \div \frac{4}{1}$$

$$\frac{3}{5} \cdot \frac{1}{4}$$

We do the same thing when we divide rational expressions.

EXAMPLE 8.27

Divide: $\frac{a^2-b^2}{3ab} \div (a^2+2ab+b^2)$.

Solution

$$\frac{a^2-b^2}{3ab} \div \left(a^2+2ab+b^2\right)$$

Write the second expression as a fraction.

$$\frac{a^2-b^2}{3ab} \div \frac{a^2+2ab+b^2}{1}$$

Rewrite the division as the fir t expression times the reciprocal of the second expression.

$$\frac{a^2-b^2}{3ab} \cdot \frac{1}{a^2+2ab+b^2}$$

Factor the numerators and the denominators, and then multiply.

$$\frac{(a-b)(a+b)\cdot 1}{3ab\cdot(a+b)(a+b)}$$

Simplify by dividing out common factors.

$$\frac{(a-b)\cancel{(a+b)}}{3ab\cdot\cancel{(a+b)}(a+b)}$$

Simplify.

$$\frac{(a-b)}{3ab(a+b)}$$

TRY IT : : 8.53 Divide: $\frac{2x^2-14x-16}{4} \div (x^2+2x+1)$.

TRY IT : : 8.54 Divide: $\frac{y^2-6y+8}{y^2-4y} \div (3y^2-12y)$.

Remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 8.28

Divide: $\dfrac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of fir t times reciprocal of second. $\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\frac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$

Simplify. $\frac{3x-2}{4}$

TRY IT : : 8.55

Divide: $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

TRY IT : : 8.56

Divide: $\frac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then we factor and multiply.

EXAMPLE 8.29

Divide: $\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators, and then multiply.	$\frac{3 \cdot 8(x-2)(x+3)(x-1)(x+2)}{4 \cdot 2(x-1)(x+2)(x-5)(x+6)}$
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4} \cdot \cancel{2}\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

TRY IT : : 8.57

Divide: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

TRY IT : : 8.58

Divide: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

8.2 EXERCISES

Practice Makes Perfect

Multiply Rational Expressions

In the following exercises, multiply.

73. $\frac{12}{16} \cdot \frac{4}{10}$

74. $\frac{32}{5} \cdot \frac{16}{24}$

75. $\frac{18}{10} \cdot \frac{4}{30}$

76. $\frac{21}{36} \cdot \frac{45}{24}$

77. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

78. $\frac{8w^3y}{9y^2} \cdot \frac{3y}{4w^4}$

79. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

80. $\frac{4mn^2}{5n^3} \cdot \frac{mn^3}{8m^2n^2}$

81. $\frac{5p^2}{p^2 - 5p - 36} \cdot \frac{p^2 - 16}{10p}$

82. $\frac{3q^2}{q^2 + q - 6} \cdot \frac{q^2 - 9}{9q}$

83. $\frac{4r}{r^2 - 3r - 10} \cdot \frac{r^2 - 25}{8r^2}$

84. $\frac{s}{s^2 - 9s + 14} \cdot \frac{s^2 - 49}{7s^2}$

85. $\frac{x^2 - 7x}{x^2 + 6x + 9} \cdot \frac{x + 3}{4x}$

86. $\frac{2y^2 - 10y}{y^2 + 10y + 25} \cdot \frac{y + 5}{6y}$

87. $\frac{z^2 + 3z}{z^2 - 3z - 4} \cdot \frac{z - 4}{z^2}$

88. $\frac{2a^2 + 8a}{a^2 - 9a + 20} \cdot \frac{a - 5}{a^2}$

89. $\frac{28 - 4b}{3b - 3} \cdot \frac{b^2 + 8b - 9}{b^2 - 49}$

90. $\frac{18c - 2c^2}{6c + 30} \cdot \frac{c^2 + 7c + 10}{c^2 - 81}$

91. $\frac{35d - 7d^2}{d^2 + 7d} \cdot \frac{d^2 + 12d + 35}{d^2 - 25}$

92. $\frac{72m - 12m^2}{8m + 32} \cdot \frac{m^2 + 10m + 24}{m^2 - 36}$

93. $\frac{4n + 20}{n^2 + n - 20} \cdot \frac{n^2 - 16}{4n + 16}$

94. $\frac{6p^2 - 6p}{p^2 + 7p - 18} \cdot \frac{p^2 - 81}{3p^2 - 27p}$

95. $\frac{q^2 - 2q}{q^2 + 6q - 16} \cdot \frac{q^2 - 64}{q^2 - 8q}$

96. $\frac{2r^2 - 2r}{r^2 + 4r - 5} \cdot \frac{r^2 - 25}{2r^2 - 10r}$

Divide Rational Expressions

In the following exercises, divide.

97. $\frac{t - 6}{3 - t} \div \frac{t^2 - 9}{t - 5}$

98. $\frac{v - 5}{11 - v} \div \frac{v^2 - 25}{v - 11}$

99. $\frac{10 + w}{w - 8} \div \frac{100 - w^2}{8 - w}$

100. $\frac{7 + x}{x - 6} \div \frac{49 - x^2}{x + 6}$

101. $\frac{27y^2}{3y - 21} \div \frac{3y^2 + 18}{y^2 + 13y + 42}$

102. $\frac{24z^2}{2z - 8} \div \frac{4z - 28}{z^2 - 11z + 28}$

103. $\frac{16a^2}{4a + 36} \div \frac{4a^2 - 24a}{a^2 + 4a - 45}$

104. $\frac{24b^2}{2b - 4} \div \frac{12b^2 + 36b}{b^2 - 11b + 18}$

105. $\frac{5c^2 + 9c + 2}{c^2 - 25} \div \frac{3c^2 - 14c - 5}{c^2 + 10c + 25}$

106. $\frac{2d^2 + d - 3}{d^2 - 16} \div \frac{2d^2 - 9d - 18}{d^2 - 8d + 16}$

107. $\dfrac{6m^2 - 2m - 10}{9 - m^2} \div \dfrac{6m^2 + 29m - 20}{m^2 - 6m + 9}$

108. $\dfrac{2n^2 - 3n - 14}{25 - n^2} \div \dfrac{2n^2 - 13n + 21}{n^2 - 10n + 25}$

109. $\dfrac{3s^2}{s^2 - 16} \div \dfrac{s^3 - 4s^2 + 16s}{s^3 - 64}$

110. $\dfrac{r^2 - 9}{15} \div \dfrac{r^3 - 27}{5r^2 + 15r + 45}$

111. $\dfrac{p^3 + q^3}{3p^2 + 3pq + 3q^2} \div \dfrac{p^2 - q^2}{12}$

112. $\dfrac{v^3 - 8w^3}{2v^2 + 4vw + 8w^2} \div \dfrac{v^2 - 4w^2}{4}$

113. $\dfrac{t^2 - 9}{2t} \div (t^2 - 6t + 9)$

114. $\dfrac{x^2 + 3x - 10}{4x} \div (2x^2 + 20x + 50)$

115. $\dfrac{2y^2 - 10yz - 48z^2}{2y - 1} \div (4y^2 - 32yz)$

116. $\dfrac{2m^2 - 98n^2}{2m + 6} \div (m^2 - 7mn)$

117. $\dfrac{\frac{2a^2 - a - 21}{5a + 20}}{\frac{a^2 + 7a + 12}{a^2 + 8a + 16}}$

118. $\dfrac{\frac{3b^2 + 2b - 8}{12b + 18}}{\frac{3b^2 + 2b - 8}{2b^2 - 7b - 15}}$

119. $\dfrac{\frac{12c^2 - 12}{2c^2 - 3c + 1}}{\frac{4c + 4}{6c^2 - 13c + 5}}$

120. $\dfrac{\frac{4d^2 + 7d - 2}{35d + 10}}{\frac{d^2 - 4}{7d^2 - 12d - 4}}$

121. $\dfrac{10m^2 + 80m}{3m - 9} \cdot \dfrac{m^2 + 4m - 21}{m^2 - 9m + 20} \div \dfrac{5m^2 + 10m}{2m - 10}$

122. $\dfrac{4n^2 + 32n}{3n + 2} \cdot \dfrac{3n^2 - n - 2}{n^2 + n - 30} \div \dfrac{108n^2 - 24n}{n + 6}$

123. $\dfrac{12p^2 + 3p}{p + 3} \div \dfrac{p^2 + 2p - 63}{p^2 - p - 12} \cdot \dfrac{p - 7}{9p^3 - 9p^2}$

124. $\dfrac{6q + 3}{9q^2 - 9q} \div \dfrac{q^2 + 14q + 33}{q^2 + 4q - 5} \cdot \dfrac{4q^2 + 12q}{12q + 6}$

Everyday Math

125. Probability The director of large company is interviewing applicants for two identical jobs. If $w =$ the number of women applicants and $m =$ the number of men applicants, then the probability that two women are selected for the jobs is $\dfrac{w}{w + m} \cdot \dfrac{w - 1}{w + m - 1}$.

ⓐ Simplify the probability by multiplying the two rational expressions.

ⓑ Find the probability that two women are selected when $w = 5$ and $m = 10$.

126. Area of a triangle The area of a triangle with base b and height h is $\dfrac{bh}{2}$. If the triangle is stretched to make a new triangle with base and height three times as much as in the original triangle, the area is $\dfrac{9bh}{2}$. Calculate how the area of the new triangle compares to the area of the original triangle by dividing $\dfrac{9bh}{2}$ by $\dfrac{bh}{2}$.

Writing Exercises

127.

ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

ⓒ Evaluate your answer to part (b) when $n = 7$. Did you get the same answer you got in part (a)? Why or why not?

128.

ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.

ⓑ Divide $\frac{x^2 - 1}{x} \div (x + 1)$ and explain all your steps.

ⓒ Evaluate your answer to part (b) when $x = 5$. Did you get the same answer you got in part (a)? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
multiply rational expressions.			
divide rational expressions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.3 Add and Subtract Rational Expressions with a Common Denominator

Learning Objectives

By the end of this section, you will be able to:

- Add rational expressions with a common denominator
- Subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{y}{3}+\frac{9}{3}$.
   If you missed this problem, review **Example 1.77**.
2. Subtract: $\frac{10}{x}-\frac{2}{x}$.
   If you missed this problem, review **Example 1.79**.
3. Factor completely: $8n^5-20n^3$.
   If you missed this problem, review **Example 7.59**.
4. Factor completely: $45a^3-5ab^2$.
   If you missed this problem, review **Example 7.62**.

Add Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition

If $p,\ q,\ \text{and}\ r$ are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r}$$

To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.

We will add two numerical fractions first, to remind us of how this is done.

EXAMPLE 8.30

Add: $\frac{5}{18}+\frac{7}{18}$.

Solution

$$\frac{5}{18}+\frac{7}{18}$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator. $\frac{5+7}{18}$

Add in the numerator. $\frac{12}{18}$

Factor the numerator and denominator to show the common factors. $\frac{6\cdot 2}{6\cdot 3}$

Remove common factors. $\frac{\cancel{6}\cdot 2}{\cancel{6}\cdot 3}$

Simplify. $\frac{2}{3}$

TRY IT : : 8.59 Add: $\frac{7}{16}+\frac{5}{16}$.

TRY IT : : 8.60 Add: $\frac{3}{10}+\frac{1}{10}$.

Remember, we do not allow values that would make the denominator zero. What value of y should be excluded in the next example?

EXAMPLE 8.31

Add: $\frac{3y}{4y-3}+\frac{7}{4y-3}$.

Solution

$$\frac{3y}{4y-3}+\frac{7}{4y-3}$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator. $\frac{3y+7}{4y-3}$

The numerator and denominator cannot be factored. The fraction is simplified.

TRY IT : : 8.61 Add: $\frac{5x}{2x+3}+\frac{2}{2x+3}$.

TRY IT : : 8.62 Add: $\frac{x}{x-2}+\frac{1}{x-2}$.

EXAMPLE 8.32

Add: $\frac{7x+12}{x+3}+\frac{x^2}{x+3}$.

Solution

	$\frac{7x+12}{x+3}+\frac{x^2}{x+3}$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{7x+12+x^2}{x+3}$
Write the degrees in descending order.	$\frac{x^2+7x+12}{x+3}$
Factor the numerator.	$\frac{(x+3)(x+4)}{x+3}$
Simplify by removing common factors.	$\frac{\cancel{(x+3)}(x+4)}{\cancel{x+3}}$
Simplify.	$x+4$

TRY IT : : 8.63 Add: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

TRY IT : : 8.64 Add: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Subtract Rational Expressions with a Common Denominator

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator.

Rational Expression Subtraction

If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

EXAMPLE 8.33

Subtract: $\frac{n^2}{n-10}-\frac{100}{n-10}$.

Solution

$$\frac{n^2}{n-10} - \frac{100}{n-10}$$

The fractions have a common denominator, so subtract the numerators and place the diffe ence over the common denominator.

$$\frac{n^2 - 100}{n-10}$$

Factor the numerator.

$$\frac{(n-10)(n+10)}{n-10}$$

Simplify by removing common factors.

$$\frac{\cancel{(n-10)}(n+10)}{\cancel{n-10}}$$

Simplify.

$$n+10$$

TRY IT : : 8.65 Subtract: $\frac{x^2}{x+3} - \frac{9}{x+3}$.

TRY IT : : 8.66 Subtract: $\frac{4x^2}{2x-5} - \frac{25}{2x-5}$.

Be careful of the signs when you subtract a binomial!

EXAMPLE 8.34

Subtract: $\frac{y^2}{y-6} - \frac{2y+24}{y-6}$.

Solution

$$\frac{y^2}{y-6} - \frac{2y+24}{y-6}$$

The fractions have a common denominator, so subtract the numerators and place the diffe ence over the common denominator.

$$\frac{y^2 - (2y+24)}{y-6}$$

Distribute the sign in the numerator.

$$\frac{y^2 - 2y - 24}{y-6}$$

Factor the numerator.

$$\frac{(y-6)(y+4)}{y-6}$$

Remove common factors.

$$\frac{\cancel{(y-6)}(y+4)}{\cancel{y-6}}$$

Simplify.

$$y+4$$

TRY IT : : 8.67 Subtract: $\frac{n^2}{n-4} - \frac{n+12}{n-4}$.

TRY IT : : 8.68

Subtract: $\frac{y^2}{y-1} - \frac{9y-8}{y-1}$.

EXAMPLE 8.35

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18} - \frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the diffe ence over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
Simplify.	$\frac{(x-2)}{(x+3)}$

TRY IT : : 8.69

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2} - \frac{3x^2+x-3}{x^2-3x+2}$.

TRY IT : : 8.70

Subtract: $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$.

Add and Subtract Rational Expressions whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d} + \frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d} + \frac{(-1)5}{(-1)(-d)}$

The denominators are the same.	$\frac{7}{d} + \frac{-5}{d}$
Simplify.	$\frac{2}{d}$

EXAMPLE 8.36

Add: $\frac{4u-1}{3u-1} + \frac{u}{1-3u}$.

Solution

	$\frac{4u-1}{3u-1} + \frac{u}{1-3u}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{4u-1}{3u-1} + \frac{(-1)u}{(-1)(1-3u)}$
Simplify the second fraction.	$\frac{4u-1}{3u-1} + \frac{-u}{3u-1}$
The denominators are the same. Add the numerators.	$\frac{4u-1-u}{3u-1}$
Simplify.	$\frac{3u-1}{3u-1}$
Simplify.	1

TRY IT : : 8.71 Add: $\frac{8x-15}{2x-5} + \frac{2x}{5-2x}$.

TRY IT : : 8.72 Add: $\frac{6y^2+7y-10}{4y-7} + \frac{2y^2+2y+11}{7-4y}$.

EXAMPLE 8.37

Subtract: $\frac{m^2-6m}{m^2-1} - \frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1} - \frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1} - \frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1} - \frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$

Distribute.	$\dfrac{m^2 - 6m + 3m + 2}{m^2 - 1}$
Combine like terms.	$\dfrac{m^2 - 3m + 2}{m^2 - 1}$
Factor the numerator and denominator.	$\dfrac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by removing common factors.	$\dfrac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\dfrac{m-2}{m+1}$

TRY IT : : 8.73 Subtract: $\dfrac{y^2 - 5y}{y^2 - 4} - \dfrac{6y - 6}{4 - y^2}$.

TRY IT : : 8.74 Subtract: $\dfrac{2n^2 + 8n - 1}{n^2 - 1} - \dfrac{n^2 - 7n - 1}{1 - n^2}$.

8.3 EXERCISES

Practice Makes Perfect

Add Rational Expressions with a Common Denominator

In the following exercises, add.

129. $\frac{2}{15}+\frac{7}{15}$

130. $\frac{4}{21}+\frac{3}{21}$

131. $\frac{7}{24}+\frac{11}{24}$

132. $\frac{7}{36}+\frac{13}{36}$

133. $\frac{3a}{a-b}+\frac{1}{a-b}$

134. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

135. $\frac{d}{d+8}+\frac{5}{d+8}$

136. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

137. $\frac{p^2+10p}{p+2}+\frac{16}{p+2}$

138. $\frac{q^2+12q}{q+3}+\frac{27}{q+3}$

139. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

140. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

141. $\frac{8t^2}{t+4}+\frac{32t}{t+4}$

142. $\frac{6v^2}{v+5}+\frac{30v}{v+5}$

143. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

144. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

145. $\frac{y^2}{y+8}-\frac{64}{y+8}$

146. $\frac{z^2}{z+2}-\frac{4}{z+2}$

147. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

148. $\frac{25b^2}{5b-6}-\frac{36}{5b-6}$

149. $\frac{c^2}{c-8}-\frac{6c+16}{c-8}$

150. $\frac{d^2}{d-9}-\frac{6d+27}{d-9}$

151. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

152. $\frac{2n^2}{4n-32}-\frac{30n-16}{4n-32}$

153. $\frac{6p^2+3p+4}{p^2+4p-5}-\frac{5p^2+p+7}{p^2+4p-5}$

154. $\frac{5q^2+3q-9}{q^2+6q+8}-\frac{4q^2+9q+7}{q^2+6q+8}$

155. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2-5r-30}{r^2-49}$

156. $\frac{7t^2-t-4}{t^2-25}-\frac{6t^2+2t-1}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add.

157. $\frac{10v}{2v-1}+\frac{2v+4}{1-2v}$

158. $\frac{20w}{5w-2}+\frac{5w+6}{2-5w}$

159. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

160. $\dfrac{6y^2 + 2y - 11}{3y - 7} + \dfrac{3y^2 - 3y + 17}{7 - 3y}$

In the following exercises, subtract.

161. $\dfrac{z^2 + 6z}{z^2 - 25} - \dfrac{3z + 20}{25 - z^2}$

162. $\dfrac{a^2 + 3a}{a^2 - 9} - \dfrac{3a - 27}{9 - a^2}$

163. $\dfrac{2b^2 + 30b - 13}{b^2 - 49} - \dfrac{2b^2 - 5b - 8}{49 - b^2}$

164. $\dfrac{c^2 + 5c - 10}{c^2 - 16} - \dfrac{c^2 - 8c - 10}{16 - c^2}$

Everyday Math

165. Sarah ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. If r represents Sarah's speed when she ran, then her running time is modeled by the expression $\frac{8}{r}$ and her biking time is modeled by the expression $\frac{24}{r+4}$. Add the rational expressions $\frac{8}{r} + \frac{24}{r+4}$ to get an expression for the total amount of time Sarah ran and biked.

166. If Pete can paint a wall in p hours, then in one hour he can paint $\frac{1}{p}$ of the wall. It would take Penelope 3 hours longer than Pete to paint the wall, so in one hour she can paint $\frac{1}{p+3}$ of the wall. Add the rational expressions $\frac{1}{p} + \frac{1}{p+3}$ to get an expression for the part of the wall Pete and Penelope would paint in one hour if they worked together.

Writing Exercises

167. Donald thinks that $\frac{3}{x} + \frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

168. Explain how you find the Least Common Denominator of $x^2 + 5x + 4$ and $x^2 - 16$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
add rational expressions with a common denominator.			
subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.4 Add and Subtract Rational Expressions with Unlike Denominators

Learning Objectives

By the end of this section, you will be able to:

- Find the least common denominator of rational expressions
- Find equivalent rational expressions
- Add rational expressions with different denominators
- Subtract rational expressions with different denominators

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
   If you missed this problem, review **Example 1.81**.
2. Subtract: $6(2x+1)-4(x-5)$.
   If you missed this problem, review **Example 1.139**.
3. Find the Greatest Common Factor of $9x^2y^3$ and $12xy^5$.
   If you missed this problem, review **Example 7.3**.
4. Factor completely $-48n-12$.
   If you missed this problem, review **Example 7.11**.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at the example $\frac{7}{12}+\frac{5}{18}$ from **Foundations**. Since the denominators are not the same, the first step was to find the least common denominator (LCD). Remember, the LCD is the least common multiple of the denominators. It is the smallest number we can use as a common denominator.

To find the LCD of 12 and 18, we factored each number into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

$$\begin{aligned} 12 &= 2\cdot 2\cdot 3 \\ 18 &= 2\cdot 3\cdot 3 \\ \hline \text{LCD} &= 2\cdot 2\cdot 3\cdot 3 \\ \text{LCD} &= 36 \end{aligned}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

HOW TO : : FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

Step 1. Factor each expression completely.

Step 2. List the factors of each expression. Match factors vertically when possible.

Step 3. Bring down the columns.

Step 4. Multiply the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

EXAMPLE 8.38

Find the LCD for $\frac{8}{x^2 - 2x - 3}, \frac{3x}{x^2 + 4x + 3}$.

Solution

$$\text{Find the LCD for } \frac{8}{x^2 - 2x - 3}, \frac{3x}{x^2 + 4x + 3}.$$

Factor each expression completely, lining up common factors. Bring down the columns.	$x^2 - 2x - 3 = (x + 1)(x - 2)$ $x^2 + 4x + 3 = (x + 1)\quad(x + 3)$ $\text{LCD} = (x + 1)(x - 2)(x + 3)$
Multiply the factors.	The LCD is $(x + 1)(x - 3)(x + 3)$.

TRY IT : : 8.75 Find the LCD for $\frac{2}{x^2 - x - 12}, \frac{1}{x^2 - 16}$.

TRY IT : : 8.76 Find the LCD for $\frac{x}{x^2 + 8x + 15}, \frac{5}{x^2 + 9x + 18}$.

Find Equivalent Rational Expressions

When we add numerical fractions, once we find the LCD, we rewrite each fraction as an equivalent fraction with the LCD.

$$\frac{7}{12} + \frac{5}{18}$$

$$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$$

$$\frac{21}{36} + \frac{10}{36}$$

$$\begin{aligned} 12 &= 2 \cdot 2 \cdot 3 \\ 18 &= 2 \cdot \quad\; 3 \cdot 3 \\ \text{LCD} &= 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCD} &= 36 \end{aligned}$$

We will do the same thing for rational expressions.

EXAMPLE 8.39

Rewrite as equivalent rational expressions with denominator $(x + 1)(x - 3)(x + 3)$: $\frac{8}{x^2 - 2x - 3}, \frac{3x}{x^2 + 4x + 3}$.

Solution

	$\frac{8}{x^2 - 2x - 3}, \frac{3x}{x^2 + 4x + 3}$
Factor each denominator.	$\frac{8}{(x + 1)(x - 3)}, \frac{3x}{(x + 1)(x + 3)}$
Find the LCD. $x^2 - 2x - 3 = (x + 1)(x - 3)$ $x^2 + 4x + 3 = (x + 1)\quad(x + 3)$ $\text{LCD} \quad = (x + 1)(x - 3)(x + 3)$	
Multiply each denominator by the 'missing' factor and multiply each numerator by the same factor.	$\frac{8(x + 3)}{(x + 1)(x - 3)(x + 3)}, \frac{3x(x - 3)}{(x + 1)(x + 3)(x - 3)}$
Simplify the numerators.	$\frac{8x + 24}{(x + 1)(x - 3)(x + 3)}, \frac{3x^2 - 9x}{(x + 1)(x + 3)(x - 3)}$

TRY IT : : 8.77 Rewrite as equivalent rational expressions with denominator $(x+3)(x-4)(x+4)$:

$\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$.

TRY IT : : 8.78 Rewrite as equivalent rational expressions with denominator $(x+3)(x+5)(x+6)$:

$\frac{x}{x^2+8x+15}, \frac{5}{x^2+9x+18}$.

Add Rational Expressions with Different Denominators

Now we have all the steps we need to add rational expressions with different denominators. As we have done previously, we will do one example of adding numerical fractions first.

EXAMPLE 8.40

Add: $\frac{7}{12}+\frac{5}{18}$.

Solution

		$\frac{7}{12}+\frac{5}{18}$
Find the LCD of 12 and 18.	$12 = 2\cdot2\cdot3$ $18 = 2\cdot\quad 3\cdot3$ $\text{LCD} = 2\cdot2\cdot3\cdot3$ $\text{LCD} = 36$	
Rewrite each fraction as an equivalent fraction with the LCD.		$\frac{7\cdot3}{12\cdot3}+\frac{5\cdot2}{18\cdot2}$
Add the fractions.		$\frac{21}{36}+\frac{10}{36}$
The fraction cannot be simplified.		$\frac{31}{36}$

TRY IT : : 8.79 Add: $\frac{11}{30}+\frac{7}{12}$.

TRY IT : : 8.80 Add: $\frac{3}{8}+\frac{9}{20}$.

Now we will add rational expressions whose denominators are monomials.

EXAMPLE 8.41

Add: $\frac{5}{12x^2y}+\frac{4}{21xy^2}$.

Solution

	$\dfrac{5}{12x^2y}+\dfrac{4}{21xy^2}$
Find the LCD of $12x^2y$ and $21xy^2$.	$12x^2y = 2\cdot2\cdot3\cdot\quad x\cdot x\cdot y$ $21xy^2 = \quad 3\cdot7\cdot x\cdot\quad y\cdot y$ $\text{LCD} = 2\cdot2\cdot3\cdot7\cdot x\cdot x\cdot y\cdot y$ $\text{LCD} = 84x^2y^2$
	$\dfrac{5}{12x^2y}+\dfrac{4}{21xy^2}$
Rewrite each rational expression as an equivalent fraction with the LCD.	$\dfrac{5\cdot7y}{12x^2y\cdot7y}+\dfrac{4\cdot4x}{21xy^2\cdot4x}$
Simplify.	$\dfrac{35y}{84x^2y^2}+\dfrac{16x}{84x^2y^2}$
Add the rational expressions.	$\dfrac{16x+35y}{84x^2y^2}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

TRY IT : : 8.81 Add: $\dfrac{2}{15a^2b}+\dfrac{5}{6ab^2}$.

TRY IT : : 8.82 Add: $\dfrac{5}{16c}+\dfrac{3}{8cd^2}$.

Now we are ready to tackle polynomial denominators.

EXAMPLE 8.42 HOW TO ADD RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Add: $\dfrac{3}{x-3}+\dfrac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3)$, $(x-2)$ Change into equivalent rational expressions with the LCD, $(x-3)(x-2)$. Keep the denominators factored!	$x-3 \;:(x-3)$ $x-2 \;:\quad (x-2)$ $\text{LCD} \;:(x-3)(x-2)$ $\dfrac{3}{x-3}+\dfrac{2}{x-2}$ $\dfrac{3(x-2)}{(x-3)(x-2)}+\dfrac{2(x-3)}{(x-2)(x-3)}$ $\dfrac{3x-6}{(x-3)(x-2)}+\dfrac{2x-6}{(x-2)(x-3)}$

Step 2. Add the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$ $\frac{5x-12}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	

TRY IT : : 8.83 Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

TRY IT : : 8.84 Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

The steps to use to add rational expressions are summarized in the following procedure box.

HOW TO : : ADD RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add the rational expressions.

Step 3. Simplify, if possible.

EXAMPLE 8.43

Add: $\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$.

Solution

	$\frac{2a}{2ab+b^2}+\frac{3a}{4a^2-b^2}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} 2ab+b^2 &= b(2a+b) \\ 4a^2-b^2 &= (2a+b)(2a-b) \\ \text{LCD} &= b(2a+b)(2a-b) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2a(2a-b)}{b(2a+b)(2a-b)}+\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$
Simplify the numerators.	$\frac{4a^2-2ab}{b(2a+b)(2a-b)}+\frac{3ab}{b(2a+b)(2a-b)}$
Add the rational expressions.	$\frac{4a^2-2ab+3ab}{b(2a+b)(2a-b)}$
Simplify the numerator.	$\frac{4a^2+ab}{b(2a+b)(2a-b)}$
Factor the numerator.	$\frac{a(4a+b)}{b(2a+b)(2a-b)}$
There are no factors common to the numerator and denominator. The fraction cannot be simplified.	

TRY IT : : 8.85 Add: $\frac{5x}{xy-y^2}+\frac{2x}{x^2+y^2}$.

TRY IT : : 8.86 Add: $\frac{7}{2m+6}+\frac{4}{m^2+4m+3}$.

Avoid the temptation to simplify too soon! In the example above, we must leave the first rational expression as $\frac{2a(2a-b)}{b(2a+b)(2a-b)}$ to be able to add it to $\frac{3a\cdot b}{(2a+b)(2a-b)\cdot b}$. Simplify only after you have combined the numerators.

EXAMPLE 8.44

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} x^2-2x-3 &= (x+1)(x-3) \\ x^2+4x+3 &= (x+1)\quad(x+3) \\ \text{LCD} &= (x+1)(x-3)(x+3) \end{aligned}$	
Rewrite each rational expression as an equivalent fraction with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$

Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)} + \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2+9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x^2+24}{(x+1)(x-3)(x+3)}$
The numerator is prime, so there are no common factors.	

TRY IT : : 8.87 Add: $\frac{1}{m^2-m-2} + \frac{5m}{m^2+3m+2}$.

TRY IT : : 8.88 Add: $\frac{2n}{n^2-3n-10} + \frac{6}{n^2+5n+6}$.

Subtract Rational Expressions with Different Denominators

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

EXAMPLE 8.45 HOW TO SUBTRACT RATIONAL EXPRESSIONS WITH DIFFERENT DENOMINATORS

Subtract: $\frac{x}{x-3} - \frac{x-2}{x+3}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes – Go to step 2. • No – Rewrite each rational expression with the LCD. ○ Find the LCD. ○ Rewrite each rational expression as an equivalent rational expression with the LCD.	No Find the LCD of $(x-3)$, $(x+3)$. Change into equivalent fractions with the LCD, $(x-3)(x+3)$. Keep the denominators factored!	$x-3$: $(x-3)$ $x+3$: $(x+3)$ LCD: $(x-3)(x+3)$ $\frac{x}{x-3} - \frac{x-2}{x+3}$ $\frac{x(x+3)}{(x-3)(x+3)} - \frac{(x-2)(x-3)}{(x+3)(x-3)}$ $\frac{x^2+3x}{(x-3)(x+3)} - \frac{x^2-5x+6}{(x-3)(x+3)}$
Step 2. Subtract the rational expressions.	Subtract the numerators and place the difference over the common denominator. Be careful with the signs!	$\frac{x^2+3x-(x^2-5x+6)}{(x-3)(x+3)}$ $\frac{x^2+3x-x^2+5x-6}{(x-3)(x+3)}$ $\frac{8x-6}{(x-3)(x+3)}$
Step 3. Simplify, if possible.	The numerator and denominator have no factors in common. The answer is simplified.	$\frac{2(4x-3)}{(x-3)(x+3)}$

TRY IT : : 8.89 Subtract: $\frac{y}{y+4} - \frac{y-2}{y-5}$.

TRY IT : : 8.90 Subtract: $\frac{z+3}{z+2}-\frac{z}{z+3}$.

The steps to take to subtract rational expressions are listed below.

HOW TO : : SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if they have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Subtract the rational expressions.

Step 3. Simplify, if possible.

EXAMPLE 8.46

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{l} y^2-16 = (y-4)(y+4) \\ y-4 \quad = y-4 \\ \hline \text{LCD} \quad = (y-4)(y+4) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors.	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

TRY IT : : 8.91 Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

TRY IT : : 8.92 Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

There are lots of negative signs in the next example. Be extra careful!

EXAMPLE 8.47

Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will mutliply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Simplify.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the expressions have a common denominator? No.	
Find the LCD. $n^2+n-6=(n-2)(n+3)$; $n-2=(n-2)$; $\text{LCD}=(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Simplify the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Somplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

TRY IT : : 8.93 Subtract: $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

TRY IT : : 8.94 Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

When one expression is not in fraction form, we can write it as a fraction with denominator 1.

EXAMPLE 8.48

Subtract: $\frac{5c+4}{c-2} - 3$.

Solution

	$\frac{5c+4}{c-2} - 3$
Write 3 as $\frac{3}{1}$ to have 2 rational expressions.	$\frac{5c+4}{c-2} - \frac{3}{1}$
Do the rational expressions have a common denominator? No.	
Find the LCD of $c-2$ and 1. LCD = $c-2$.	
Rewrite $\frac{3}{1}$ as an equivalent rational expression with the LCD.	$\frac{5c+4}{c-2} - \frac{3(c-2)}{1(c-2)}$
Simplify.	$\frac{5c+4}{c-2} - \frac{3c-6}{c-2}$
Subtract the rational expressions.	$\frac{5c+4-(3c-6)}{c-2}$
Simplify.	$\frac{2c+10}{c-2}$
Factor to check for common factors.	$\frac{2(c+5)}{c-2}$
There are no common factors; the rational expression is simplified.	

TRY IT : : 8.95 Subtract: $\frac{2x+1}{x-7} - 3$.

TRY IT : : 8.96 Subtract: $\frac{4y+3}{2y-1} - 5$.

HOW TO : : ADD OR SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.
Find the LCD.
Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example we will start by factoring all three denominators to find their LCD.

EXAMPLE 8.49

Simplify: $\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1} + \frac{1}{u} - \frac{2u-1}{u^2-u}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{l} u-1 = \quad u-1 \\ u \quad\;\; = u \\ u^2-u = u(u-1) \\ \hline \text{LCD} = u(u-1) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2 + u - 1 - 2u + 1}{u(u-1)}$
Simplify.	$\frac{2u^2 - u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

TRY IT : : 8.97 Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

TRY IT : : 8.98 Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

8.4 EXERCISES

Practice Makes Perfect

In the following exercises, find the LCD.

169. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

170. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

171. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

172. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

173. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

174. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-10c+16}$

175. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

176. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

In the following exercises, write as equivalent rational expressions with the given LCD.

177. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$
LCD $(x-4)(x+2)(x+3)$

178. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$
LCD $(y+7)(y+5)(y-6)$

179. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$
LCD $(z-2)(z+4)(z+2)$

180. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$
LCD $(a+9)(a+5)(a-9)$

181. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$
LCD $(b+3)(b+3)(b-5)$

182. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-10c+10}$
LCD $(c-2)(c-2)(c-8)$

183. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$
LCD $(3d-1)(d+5)(d-6)$

184. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$
LCD $(5m+2)(m-1)(m+3)$

In the following exercises, add.

185. $\frac{5}{24}+\frac{11}{36}$

186. $\frac{7}{30}+\frac{13}{45}$

187. $\frac{9}{20}+\frac{11}{30}$

188. $\frac{8}{27}+\frac{7}{18}$

189. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

190. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

191. $\frac{1}{2m}+\frac{7}{8m^2n}$

192. $\frac{5}{6p^2q}+\frac{1}{4p}$

193. $\frac{3}{r+4}+\frac{2}{r-5}$

194. $\frac{4}{s-7}+\frac{5}{s+3}$

195. $\frac{8}{t+5}+\frac{6}{t-5}$

196. $\frac{7}{v+5}+\frac{9}{v-5}$

197. $\frac{5}{3w-2}+\frac{2}{w+1}$

198. $\frac{4}{2x+5}+\frac{2}{x-1}$

199. $\frac{2y}{y+3}+\frac{3}{y-1}$

200. $\frac{3z}{z-2}+\frac{1}{z+5}$

201. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

202. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

203. $\frac{2m}{3m-3}+\frac{5m}{m^2+3m-4}$

204. $\frac{3}{4n+4}+\frac{6}{n^2-n-2}$

205. $\frac{3}{n^2+3n-18}+\frac{4n}{n^2+8n+12}$

206. $\frac{6}{q^2-3q-10}+\frac{5q}{q^2-8q+15}$

207. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

208. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

In the following exercises, subtract.

209. $\frac{t}{t-6}-\frac{t-2}{t+6}$

210. $\frac{v}{v-3}-\frac{v-6}{v+1}$

211. $\frac{w+2}{w+4}-\frac{w}{w-2}$

212. $\frac{x-3}{x+6}-\frac{x}{x+3}$

213. $\frac{y-4}{y+1}-\frac{1}{y+7}$

214. $\frac{z+8}{z-3}-\frac{z}{z-2}$

215. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

216. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

217. $\frac{6c}{c^2-25}-\frac{3}{c+5}$

218. $\frac{4d}{d^2-81}-\frac{2}{d+9}$

219. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

220. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

221. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

222. $\frac{7q+8}{q^2-2q-24}-\frac{q+2}{4-q}$

223. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

224. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

225. $\frac{5v-2}{v+3}-4$

226. $\frac{6w+5}{w-1}+2$

227. $\frac{2x+7}{10x-1}+3$

228. $\frac{8y-4}{5y+2}-6$

In the following exercises, add and subtract.

229. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

230. $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

231. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{11c}{c^2-4}$

232. $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

In the following exercises, simplify.

233. $\frac{6a}{3ab+b^2}+\frac{3a}{9a^2-b^2}$

234. $\frac{2c}{2c+10}+\frac{7c}{c^2+9c+20}$

235. $\frac{6d}{d^2-64}-\frac{3}{d-8}$

236. $\frac{5}{n+7}-\frac{10n}{n^2-49}$

237. $\frac{4m}{m^2+6m-7}+\frac{2}{m^2+10m+21}$

238. $\frac{3p}{p^2+4p-12}+\frac{1}{p^2+p-30}$

239. $\frac{-5n-5}{n^2+n-6}+\frac{n+1}{2-n}$

240. $\frac{-4b-24}{b^2+b-30}+\frac{b+7}{5-b}$

241. $\frac{7}{15p}+\frac{5}{18pq}$

242. $\frac{3}{20a^2}+\frac{11}{12ab^2}$

243. $\frac{4}{x-2}+\frac{3}{x+5}$

244. $\frac{6}{m+4}+\frac{9}{m-8}$

245. $\frac{2q+7}{y+4} - 2$

246. $\frac{3y-1}{y+4} - 2$

247. $\frac{z+2}{z-5} - \frac{z}{z+1}$

248. $\frac{t}{t-5} - \frac{t-1}{t+5}$

249. $\frac{3d}{d+2} + \frac{4}{d} - \frac{d+8}{d^2+2d}$

250. $\frac{2q}{q+5} + \frac{3}{q-3} - \frac{13q+15}{q^2+2q-15}$

Everyday Math

251. Decorating cupcakes Victoria can decorate an order of cupcakes for a wedding in t hours, so in 1 hour she can decorate $\frac{1}{t}$ of the cupcakes. It would take her sister 3 hours longer to decorate the same order of cupcakes, so in 1 hour she can decorate $\frac{1}{t+3}$ of the cupcakes.

ⓐ Find the fraction of the decorating job that Victoria and her sister, working together, would complete in one hour by adding the rational expressions $\frac{1}{t} + \frac{1}{t+3}$.

ⓑ Evaluate your answer to part (a) when $t = 5$.

252. Kayaking When Trina kayaks upriver, it takes her $\frac{5}{3-c}$ hours to go 5 miles, where c is the speed of the river current. It takes her $\frac{5}{3+c}$ hours to kayak 5 miles down the river.

ⓐ Find an expression for the number of hours it would take Trina to kayak 5 miles up the river and then return by adding $\frac{5}{3-c} + \frac{5}{3+c}$.

ⓑ Evaluate your answer to part (a) when $c = 1$ to find the number of hours it would take Trina if the speed of the river current is 1 mile per hour.

Writing Exercises

253. Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x+y}$.

ⓐ Choose numerical values for *x* and *y* and evaluate $\frac{1}{x} + \frac{1}{y}$.

ⓑ Evaluate $\frac{2}{x+y}$ for the same values of *x* and *y* you used in part (a).

ⓒ Explain why Felipe is wrong.

ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

254. Simplify the expression $\frac{4}{n^2+6n+9} - \frac{1}{n^2-9}$ and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the least common denominator of rational expressions.			
find equivalent rational expressions.			
add rational expressions with different denominators.			
subtract rational expressions with different denominators.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.5 Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.

   If you missed this problem, review **Example 1.72**.

2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.

   If you missed this problem, review **Example 1.74**.

Complex fractions are fractions in which the numerator or denominator contains a fraction. In Chapter 1 we simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section we will simplify *complex rational expressions*, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator or denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

Simplify a Complex Rational Expression by Writing it as Division

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right)$$

Then we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify rational expressions. We write it as if we were dividing two fractions.

EXAMPLE 8.50

Simplify: $\dfrac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$.

Solution

$$\dfrac{\frac{4}{y-3}}{\frac{8}{y^2-9}}$$

Rewrite the complex fraction as division. $\quad \dfrac{4}{y-3} \div \dfrac{8}{y^2-9}$

Rewrite as the product of fir t times the reciprocal of the second. $\quad \dfrac{4}{y-3} \cdot \dfrac{y^2-9}{8}$

Multiply. $\quad \dfrac{4\left(y^2-9\right)}{8(y-3)}$

Factor to look for common factors. $\quad \dfrac{4(y-3)(y+3)}{4 \cdot 2(y-3)}$

Remove common factors. $\quad \dfrac{\cancel{4}\cancel{(y-3)}(y+3)}{\cancel{4} \cdot 2\cancel{(y-3)}}$

Simplify. $\quad \dfrac{y+3}{2}$

Are there any value(s) of y that should not be allowed? The simplified rational expression has just a constant in the denominator. But the original complex rational expression had denominators of $y-3$ and y^2-9. This expression would be undefined if $y=3$ or $y=-3$.

TRY IT : : 8.99

Simplify: $\dfrac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

TRY IT : : 8.100

Simplify: $\dfrac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

EXAMPLE 8.51

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\dfrac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$\dfrac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Simplify the numerator and denominator, again.	$\dfrac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first times by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

TRY IT : : 8.101 Simplify: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

TRY IT : : 8.102 Simplify: $\dfrac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

EXAMPLE 8.52 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Find a common denominator and subtract the fractions in the numerator.	$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\frac{1}{x-y}$

TRY IT : : 8.103

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

TRY IT : : 8.104

Simplify: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

Step 1. Simplify the numerator and denominator.

Step 2. Rewrite the complex rational expression as a division problem.

Step 3. Divide the expressions.

EXAMPLE 8.53

Simplify: $\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution

	$\dfrac{n - \frac{4n}{n+5}}{\frac{1}{n+5} + \frac{1}{n-5}}$
Simplify the numerator and denominator.	
Find the LCD and add the fractions in the numerator. Find the LCD and add the fractions in the denominator.	$\dfrac{\frac{n(n+5)}{1(n+5)} - \frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)} + \frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\frac{n^2+5n}{n+5} - \frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)} + \frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator. Simplify.	$\dfrac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\dfrac{n^2+n}{n+5} \div \dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\dfrac{n^2+n}{n+5} \cdot \dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\dfrac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\dfrac{(n+1)(n-5)}{2}$

TRY IT : : 8.105

Simplify: $\dfrac{b - \frac{3b}{b+5}}{\frac{2}{b+5} + \frac{1}{b-5}}$.

TRY IT : : 8.106

Simplify: $\dfrac{1 - \frac{3}{c+4}}{\frac{1}{c+4} + \frac{c}{3}}$.

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by LCD of all the rational expressions.

Let’s look at the complex rational expression we simplified one way in Example 8.51. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\dfrac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

EXAMPLE 8.54

Simplify: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\dfrac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\dfrac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\dfrac{2+1}{3-2}$
	$\dfrac{3}{1}$
	3

TRY IT : : 8.107 Simplify: $\dfrac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

TRY IT : : 8.108 Simplify: $\dfrac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

EXAMPLE 8.55 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD

Simplify: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is xy.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$

Step 3. Simplify the expression.	Distribute.	$\dfrac{xy \cdot \frac{1}{x} + xy \cdot \frac{1}{y}}{xy \cdot \frac{x}{y} - xy \cdot \frac{y}{x}}$ $\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$\dfrac{1}{x-y}$

TRY IT : : 8.109 Simplify: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

TRY IT : : 8.110 Simplify: $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

EXAMPLE 8.56

Simplify: $\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution

	$\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	

Distribute in the denominator.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}(2x+10)}$
Simplify.	$\dfrac{x-6}{2x+10}$
Notice that there are no more factors common to the numerator and denominator.	

TRY IT : : 8.111 Simplify: $\dfrac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

TRY IT : : 8.112 Simplify: $\dfrac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

EXAMPLE 8.57

Simplify: $\dfrac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Solution

	$\dfrac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$\dfrac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$

Simplify.	$\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$\frac{4}{3m-12-2m+6}$
Combine like terms.	$\frac{4}{m-6}$

TRY IT : : 8.113

Simplify: $\dfrac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

TRY IT : : 8.114

Simplify: $\dfrac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

EXAMPLE 8.58

Simplify: $\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution

	$\dfrac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	
The LCD is $(y+1)(y-1)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\dfrac{\cancel{(y+1)}(y-1)\left(\frac{y}{\cancel{y+1}}\right)}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$
Simplify.	$\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator, and leave the numerator factored.	$\frac{y(y-1)}{y^2-1+y+1}$
	$\frac{y(y-1)}{y^2+y}$
Factor the denominator, and remove factors common with the numerator.	$\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\frac{y-1}{y+1}$

TRY IT : : 8.115

Simplify: $\dfrac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

TRY IT : : 8.116

Simplify: $\dfrac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

8.5 EXERCISES

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

255. $\dfrac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

256. $\dfrac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

257. $\dfrac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

258. $\dfrac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

259. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

260. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

261. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

262. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

263. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

264. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

265. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

266. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

267. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

268. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}-\frac{2}{y+4}}$

269. $\dfrac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

270. $\dfrac{4-\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

271. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

272. $\dfrac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

273. $\dfrac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

274. $\dfrac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

275. $\dfrac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

276. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

277. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

278. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

279. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

280. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

281. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

282. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

283. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

284. $\dfrac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$

285. $\dfrac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$

286. $\dfrac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

287. $\dfrac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

288. $\dfrac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

289. $\dfrac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

290. $\dfrac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

Simplify

In the following exercises, use either method.

291. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

292. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

293. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

294. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

295. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

296. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

297. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

298. $\dfrac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Everyday Math

299. Electronics The resistance of a circuit formed by connecting two resistors in parallel is $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

ⓐ Simplify the complex fraction $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$.

ⓑ Find the resistance of the circuit when $R_1 = 8$ and $R_2 = 12$.

300. Ironing Lenore can do the ironing for her family's business in h hours. Her daughter would take $h+2$ hours to get the ironing done. If Lenore and her daughter work together, using 2 irons, the number of hours it would take them to do all the ironing is $\dfrac{1}{\frac{1}{h}+\frac{1}{h+2}}$.

ⓐ Simplify the complex fraction $\dfrac{1}{\frac{1}{h}+\frac{1}{h+2}}$.

ⓑ Find the number of hours it would take Lenore and her daughter, working together, to get the ironing done if $h = 4$.

Writing Exercises

301. In this section, you learned to simplify the complex fraction $\dfrac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways:

rewriting it as a division problem

multiplying the numerator and denominator by the LCD

Which method do you prefer? Why?

302. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator. Explain what is wrong with Efraim's plan.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

8.6 Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Solve a rational equation for a specific variable

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve: $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$.
   If you missed this problem, review **Example 2.48**.
2. Solve: $n^2 - 5n - 36 = 0$.
   If you missed this problem, review **Example 7.73**.
3. Solve for y in terms of x: $5x + 2y = 10$ for y.
   If you missed this problem, review **Example 2.65**.

After defining the terms *expression* and *equation* early in **Foundations**, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will solve rational equations.

The definition of a rational equation is similar to the definition of equation we used in **Foundations**.

Rational Equation

A **rational equation** is two rational expressions connected by an equal sign.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	**Rational Equation**
$\frac{1}{8}x + \frac{1}{2}$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36} = y+1$
$\frac{1}{n-3} + \frac{1}{n+4}$	$\frac{1}{n-3} + \frac{1}{n+4} = \frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to “clear” the fractions.

Here is an example we did when we worked with linear equations:

	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$	LCD = 8
We multiplied both sides by the LCD.	$8\left(\frac{1}{8}x + \frac{1}{2}\right) = 8\left(\frac{1}{4}\right)$	
Then we distributed.	$8 \cdot \frac{1}{8}x + 8 \cdot \frac{1}{2} = 8 \cdot \frac{1}{4}$	
We simplified—and then we had an equation with no fractions.	$x + 4 = 2$	
Finally, we solved that equation.	$x + 4 - 4 = 2 - 4$	
	$x = -2$	

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then we will have an equation that does not contain rational expressions and thus is much easier for us to solve.

But because the original equation may have a variable in a denominator we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an *extraneous solution*.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, *c*, by writing $x \neq c$ next to the equation.

EXAMPLE 8.59 HOW TO SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS

Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Solution

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, 6*x*.	$6x \cdot \left(\frac{1}{x} + \frac{1}{3}\right) = 6x \cdot \left(\frac{5}{6}\right)$
	Use the Distributive Property.	$6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left(\frac{5}{6}\right)$
	Simplify – and notice, no more fractions!	$6 + 2x = 5x$
Step 4. Solve the resulting equation.	Simplify.	$6 = 3x$ $2 = x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x = 2$ into the original equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$ $\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6} = \frac{5}{6}$ ✓

TRY IT : : 8.117 Solve: $\frac{1}{y} + \frac{2}{3} = \frac{1}{5}$.

TRY IT : : 8.118 Solve: $\frac{2}{3} + \frac{1}{5} = \frac{1}{x}$.

The steps of this method are shown below.

HOW TO : : SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

Step 1. Note any value of the variable that would make any denominator zero.
Step 2. Find the least common denominator of *all* denominators in the equation.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
Step 4. Solve the resulting equation.
Step 5. Check.
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

EXAMPLE 8.60

Solve: $1 - \frac{5}{y} = -\frac{6}{y^2}$.

Solution

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$
Check.	
We did not get 0 as an algebraic solution.	

Check $y=2$ and $y=3$ in the original equation.

$1-\frac{5}{y}=-\frac{6}{y^2}$	$1-\frac{5}{y}=-\frac{6}{y^2}$
$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2}$	$1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$
$1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$	$1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$
$\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$	$\frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$
$-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4}$	$-\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$
$-\frac{3}{2}=-\frac{3}{2}$ ✓	$-\frac{2}{3}=-\frac{2}{3}$ ✓

TRY IT : : 8.119 Solve: $1-\frac{2}{a}=\frac{15}{a^2}$.

TRY IT : : 8.120 Solve: $1-\frac{4}{b}=\frac{12}{b^2}$.

EXAMPLE 8.61

Solve: $\frac{5}{3u-2}=\frac{3}{2u}$.

Solution

	$\dfrac{5}{3u-2}=\dfrac{3}{2u}$
Note any value of the variable that would make any denominator zero.	$\dfrac{5}{3u-2}=\dfrac{3}{2u},\ u\neq\dfrac{2}{3},\ u\neq 0$
Find the least common denominator of all denominators in the equation. The LCD is $2u(3u-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$2u(3u-2)\left(\dfrac{5}{3u-2}\right)=2u(3u-2)\left(\dfrac{3}{2u}\right)$
Remove common factors.	$2u\cancel{(3u-2)}\left(\dfrac{5}{\cancel{3u-2}}\right)=\cancel{2u}(3u-2)\left(\dfrac{3}{\cancel{2u}}\right)$
Simplify.	$2u(5)=(3u-2)(3)$
Multiply.	$10u=9u-6$
Solve the resulting equation.	$u=-6$
We did not get 0 or $\frac{2}{3}$ as algebraic solutions.	

Check $u=-6$ in the original equation.

$$\frac{5}{3u-2}=\frac{3}{2u}$$

$$\frac{5}{3(-6)-2}\stackrel{?}{=}\frac{3}{2(-6)}$$

$$\frac{5}{-20}\stackrel{?}{=}\frac{3}{-12}$$

$$-\frac{1}{4}=-\frac{1}{4}\checkmark$$

TRY IT : : 8.121 Solve: $\dfrac{1}{x-1}=\dfrac{2}{3x}$.

TRY IT : : 8.122 Solve: $\dfrac{3}{5n+1}=\dfrac{2}{3n}$.

When one of the denominators is a quadratic, remember to factor it first to find the LCD.

EXAMPLE 8.62

Solve: $\dfrac{2}{p+2}+\dfrac{4}{p-2}=\dfrac{p-1}{p^2-4}$.

Solution

	$\dfrac{2}{p+2}+\dfrac{4}{p-2}=\dfrac{p-1}{p^2-4}$
Note any value of the variable that would make any denominator zero.	$\dfrac{2}{p+2}+\dfrac{4}{p-2}=\dfrac{p-1}{(p+2)(p+2)},\ p\neq -2,\ p\neq 2$

Find the least common denominator of all denominators in the equation. The LCD is $(p+2)(p-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(p+2)(p-2)\left(\frac{2}{p+2}+\frac{4}{p-2}\right)=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Distribute.	$(p+2)(p-2)\frac{2}{p+2}+(p+2)(p-2)\frac{4}{p-2}=(p+2)(p-2)\left(\frac{p-1}{p^2-4}\right)$
Remove common factors.	$\cancel{(p+2)}(p-2)\frac{2}{\cancel{p+2}}+(p+2)\cancel{(p-2)}\frac{4}{\cancel{p-2}}=\cancel{(p+2)(p-2)}\left(\frac{p-1}{\cancel{p^2-4}}\right)$
Simplify.	$2(p-2)+4(p+2)=p-1$
Distribute.	$2p-4+4p+8=p-1$
Solve.	$6p+4=p-1$
	$5p=-5$
	$p=-1$
We did not get $2 \text{ or } -2$ as algebraic solutions.	

Check $p=-1$ in the original equation.

$$\frac{2}{p+2}+\frac{4}{p-2}=\frac{p-1}{p^2-4}$$
$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$
$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{6}{3}-\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$
$$\frac{2}{3}=\frac{2}{3}\checkmark$$

TRY IT : : 8.123 Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

TRY IT : : 8.124 Solve: $\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$.

EXAMPLE 8.63

Solve: $\frac{4}{q-4}-\frac{3}{q-3}=1$.

Solution

	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
Note any value of the variable that would make any denominator zero.	$\frac{4}{q-4} + \frac{3}{q-3} = 1, \quad q \neq 4, q \neq 3$
Find the least common denominator of all denominators in the equation. The LCD is $(q-4)(q-3)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(q-4)(q-3)\left(\frac{4}{q-4} - \frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Distribute.	$(q-4)(q-3)\left(\frac{4}{q-4}\right) - (q-4)(q-3)\left(\frac{3}{q-3}\right) = (q-4)(q-3)(1)$
Remove common factors.	$\cancel{(q-4)}(q-3)\left(\frac{4}{\cancel{q-4}}\right) - (q-4)\cancel{(q-3)}\left(\frac{3}{\cancel{q-3}}\right) = (q-4)(q-3)(1)$
Simplify.	$4(q-3) - 3(q-4) = (q-4)(q-3)$
Simplify.	$4q - 12 - 3q + 12 = q^2 - 7q + 12$
Combine like terms.	$q = q^2 - 7q + 12$
Solve. First write in standard form.	$0 = q^2 - 8q + 12$
Factor.	$0 = (q-2)(q-6)$
Use the Zero Product Property.	$q = 2$ or $q = 6$
We did not get 4 or 3 as algebraic solutions.	

Check $q = 2$ and $q = 6$ in the original equation.

$\frac{4}{q-4} - \frac{3}{q-3} = 1$	$\frac{4}{q-4} - \frac{3}{q-3} = 1$
$\frac{4}{2-4} - \frac{3}{2-3} \stackrel{?}{=} 1$	$\frac{4}{6-4} - \frac{3}{6-3} \stackrel{?}{=} 1$
$\frac{4}{-2} - \frac{3}{-1} \stackrel{?}{=} 1$	$\frac{4}{2} - \frac{3}{1} \stackrel{?}{=} 1$
$-2 - (-3) \stackrel{?}{=} 1$	$2 - 1 \stackrel{?}{=} 1$
$1 = 1$ ✓	$1 = 1$ ✓

TRY IT : : 8.125 Solve: $\frac{2}{x+5} - \frac{1}{x-1} = 1.$

TRY IT : : 8.126 Solve: $\frac{3}{x+8} - \frac{2}{x-2} = 1.$

EXAMPLE 8.64

Solve: $\frac{m+11}{m^2 - 5m + 4} = \frac{5}{m-4} - \frac{3}{m-1}.$

Solution

	$\frac{m+11}{m^2-5m+4} = \frac{5}{m-4} = \frac{3}{m-1}$
Factor all the denominators, so we can note any value of the variable the would make any denominator zero.	$\frac{m+11}{(m-4)(m-1)} = \frac{5}{m-4} - \frac{3}{m-1},\ m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\left(\frac{5}{m-4} - \frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right) = (m-4)(m-1)\frac{5}{m-4} - (m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)(m-1)}\left(\frac{m+11}{\cancel{(m-4)(m-1)}}\right) = \cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}} - (m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m+11 = 5(m-1) - 3(m-4)$
Solve the resulting equation.	$m+11 = 5m-5-3m+12$
	$4 = m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution. There is no solution to this equation.	

TRY IT : : 8.127 Solve: $\frac{x+13}{x^2-7x+10} = \frac{6}{x-5} - \frac{4}{x-2}$.

TRY IT : : 8.128 Solve: $\frac{y-14}{y^2+3y-4} = \frac{2}{y+4} + \frac{7}{y-1}$.

The equation we solved in **Example 8.64** had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. Some equations have no solution.

EXAMPLE 8.65

Solve: $\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$.

Solution

	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n}$
Note any value of the variable that would make any denominator zero.	$\frac{n}{12} + \frac{n+3}{3n} = \frac{1}{n},\ n \neq 0$

Find the least common denominator of all denominators in the equation. The LCD is $12n$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$12n\left(\frac{n}{12}+\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$
Distribute.	$12n\left(\frac{n}{12}\right)+12n\left(\frac{n+3}{3n}\right)=12n\left(\frac{1}{n}\right)$
Remove common factors.	$\cancel{12}n\left(\frac{n}{\cancel{12}}\right)+4\cdot\cancel{3n}\left(\frac{n+3}{\cancel{3n}}\right)=12\cancel{n}\left(\frac{1}{\cancel{n}}\right)$
Simplify.	$n\cdot n+4(n+3)=12\cdot 1$
Solve the resulting equation.	$n^2+4n+12=12$
	$n^2+4n=0$
	$n(n+4)=0$
	$n=0$ or $n=-4$
Check.	
$n=0$ is an extraneous solution.	

Check $n=-4$ in the original equation.

$$\frac{n}{12}+\frac{n+3}{3n}=\frac{1}{n}$$
$$\frac{-4}{12}+\frac{-4+3}{3(-4)}\overset{?}{=}\frac{1}{-4}$$
$$-\frac{4}{12}+\frac{1}{12}\overset{?}{=}-\frac{1}{4}$$
$$-\frac{3}{12}\overset{?}{=}-\frac{1}{4}$$
$$-\frac{1}{4}=-\frac{1}{4}\checkmark$$

TRY IT : : 8.129 Solve: $\frac{x}{18}+\frac{x+6}{9x}=\frac{2}{3x}$.

TRY IT : : 8.130 Solve: $\frac{y+5}{5y}+\frac{y}{15}=\frac{1}{y}$.

EXAMPLE 8.66

Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$

Find the least common denominator. The LCD is $(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right) = (y-6)(y+6)\left(\frac{72}{(y-6)(y+6)} + 4\right)$
Simplify.	$(y-6) \cdot y = 72 + (y-6)(y+6) \cdot 4$
Simplify.	$y(y-6) = 72 + 4(y^2 - 36)$
Solve the resulting equation.	$y^2 - 6y = 72 + 4y^2 - 144$
	$0 = 3y^2 + 6y - 72$
	$0 = 3(y^2 + 2y - 24)$
	$0 = 3(y+6)(y-4)$
	$y = -6,\ y = 4$
Check.	
$y = -6$ is an extraneous solution.	

Check $y = 4$ in the original equation.

$$\frac{y}{y+6} = \frac{72}{y^2-36} + 4$$

$$\frac{4}{4+6} \stackrel{?}{=} \frac{72}{4^2-36} + 4$$

$$\frac{4}{10} \stackrel{?}{=} \frac{72}{-20} + 4$$

$$\frac{4}{10} \stackrel{?}{=} -\frac{36}{10} + \frac{40}{10}$$

$$\frac{4}{10} = \frac{4}{10} \checkmark$$

TRY IT : : 8.131 Solve: $\frac{x}{x+4} = \frac{32}{x^2-16} + 5.$

TRY IT : : 8.132 Solve: $\frac{y}{y+8} = \frac{128}{y^2-64} + 9.$

EXAMPLE 8.67

Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}.$

Solution

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator.The LCD is $12(x-1)(x+1)$	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2-2x+9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2-2x+9$
Solve the resulting equation.	$6x^2+6x-8x+8 = 5x^2-2x+9$
	$x^2-1=0$
	$(x-1)(x+1)=0$
	$x=1$ or $x=-1$
Check.	
$x=1$ and $x=-1$ are extraneous solutions. The equation has no solution.	

TRY IT : : 8.133 Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

TRY IT : : 8.134 Solve: $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-6}{8z^2+8z-64}$.

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

We'll start with a formula relating distance, rate, and time. We have used it many times before, but not usually in this form.

EXAMPLE 8.68

Solve: $\frac{D}{T} = R$ for T.

Solution

	$\frac{D}{T} = R$ for T
Note any value of the variable that would make any denominator zero.	$\frac{D}{T} = R, T \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, *T*.	$T\left(\frac{D}{T}\right) = T(R)$
Simplify.	$D = T \cdot R$
Divide both sides by *R* to isolate *T*.	$\frac{D}{R} = \frac{RT}{R}$
Simplify.	$\frac{D}{R} = T$

TRY IT : : 8.135 Solve: $\frac{A}{L} = W$ for L.

TRY IT : : 8.136 Solve: $\frac{F}{A} = M$ for A.

Example 8.69 uses the formula for slope that we used to get the point-slope form of an equation of a line.

EXAMPLE 8.69

Solve: $m = \frac{x-2}{y-3}$ for y.

Solution

	$m = \frac{x-2}{y-3}$ for y
Note any value of the variable that would make any denominator zero.	$m = \frac{x-2}{y-3},\ y \neq 3$
Clear the fractions by multiplying both sides of the equations by the LCD, $y-3$.	$(y-3)m = (y-3)\left(\frac{x-2}{y-3}\right)$
Simplify.	$ym - 3m = x - 2$
Isolate the term with *y*.	$ym = x - 2 + 3m$
Divide both sides by *m* to isolate *y*.	$\frac{ym}{m} = \frac{x-2+3m}{m}$
Simplify.	$y = \frac{x-2+3m}{m}$

TRY IT : : 8.137 Solve: $\frac{y-2}{x+1} = \frac{2}{3}$ for x.

TRY IT : : 8.138 Solve: $x = \frac{y}{1-y}$ for y.

Be sure to follow all the steps in **Example 8.70**. It may look like a very simple formula, but we cannot solve it instantly for either denominator.

EXAMPLE 8.70

Solve $\frac{1}{c}+\frac{1}{m}=1$ for c.

Solution

	$\frac{1}{c}+\frac{1}{m}=1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm.	$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$
Simplify.	$m+c=cm$
Collect the terms with c to the right.	$m=cm-c$
Factor the expression on the right.	$m=c(m-1)$
To isolate c, divide both sides by $m-1$.	$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1}=c$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

TRY IT : : 8.139 Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

TRY IT : : 8.140 Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

8.6 EXERCISES

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve.

303. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

304. $\frac{5}{6}+\frac{3}{b}=\frac{1}{3}$

305. $\frac{5}{2}-\frac{1}{c}=\frac{3}{4}$

306. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

307. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

308. $\frac{3}{7}+\frac{2}{3}=\frac{1}{w}$

309. $\frac{7}{9}+\frac{1}{x}=\frac{2}{3}$

310. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

311. $1-\frac{2}{m}=\frac{8}{m^2}$

312. $1+\frac{4}{n}=\frac{21}{n^2}$

313. $1+\frac{9}{p}=\frac{-20}{p^2}$

314. $1-\frac{7}{q}=\frac{-6}{q^2}$

315. $\frac{1}{r+3}=\frac{4}{2r}$

316. $\frac{3}{t-6}=\frac{1}{t}$

317. $\frac{5}{3v-2}=\frac{7}{4v}$

318. $\frac{8}{2w+1}=\frac{3}{w}$

319. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

320. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

321. $\frac{8}{z-10}+\frac{7}{z+10}=\frac{5}{z^2-100}$

322. $\frac{9}{a+11}+\frac{6}{a-11}=\frac{7}{a^2-121}$

323. $\frac{1}{q+4}-\frac{7}{q-2}=1$

324. $\frac{3}{r+10}-\frac{4}{r-4}=1$

325. $\frac{1}{t+7}-\frac{5}{t-5}=1$

326. $\frac{2}{s+7}-\frac{3}{s-3}=1$

327. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

328. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

329. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

330. $\frac{y-3}{y^2-4y-5}=\frac{1}{y+1}+\frac{8}{y-5}$

331. $\frac{z}{16}+\frac{z+2}{4z}=\frac{1}{2z}$

332. $\frac{a}{9}+\frac{a+3}{3a}=\frac{1}{a}$

333. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

334. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

335. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

336. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

337. $\frac{n}{n+2}=\frac{8}{m^2-4}+3$

338. $\frac{p}{p+7}=\frac{98}{p^2-49}+8$

339. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

340. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

341. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

342. $\frac{t}{6t-12} - \frac{5}{2t+10} = \frac{t^2 - 23t + 70}{12t^2 + 36t - 120}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

343. $\frac{C}{r} = 2\pi$ for r

344. $\frac{I}{r} = P$ for r

345. $\frac{V}{h} = lw$ for h

346. $\frac{2A}{b} = h$ for b

347. $\frac{v+3}{w-1} = \frac{1}{2}$ for w

348. $\frac{x+5}{2-y} = \frac{4}{3}$ for y

349. $a = \frac{b+3}{c-2}$ for c

350. $m = \frac{n}{2-n}$ for n

351. $\frac{1}{p} + \frac{2}{q} = 4$ for p

352. $\frac{3}{s} + \frac{1}{t} = 2$ for s

353. $\frac{2}{v} + \frac{1}{5} = \frac{1}{2}$ for v

354. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y

355. $\frac{m+3}{n-2} = \frac{4}{5}$ for n

356. $\frac{E}{c} = m^2$ for c

357. $\frac{3}{x} - \frac{5}{y} = \frac{1}{4}$ for y

358. $\frac{R}{T} = W$ for T

359. $r = \frac{s}{3-t}$ for t

360. $c = \frac{2}{a} + \frac{b}{5}$ for a

Everyday Math

361. House Painting Alain can paint a house in 4 days. Spiro would take 7 days to paint the same house. Solve the equation $\frac{1}{4} + \frac{1}{7} = \frac{1}{t}$ for t to find the number of days it would take them to paint the house if they worked together.

362. Boating Ari can drive his boat 18 miles with the current in the same amount of time it takes to drive 10 miles against the current. If the speed of the boat is 7 knots, solve the equation $\frac{18}{7+c} = \frac{10}{7-c}$ for c to find the speed of the current.

Writing Exercises

363. Why is there no solution to the equation $\frac{3}{x-2} = \frac{5}{x-2}$?

364. Pete thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Pete is wrong.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations for a specific variable.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.7 Solve Proportion and Similar Figure Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve proportions
- Solve similar figure applications

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Solve $\frac{n}{3} = 30$.

   If you missed this problem, review **Example 2.21**.

2. The perimeter of a triangular window is 23 feet. The lengths of two sides are ten feet and six feet. How long is the third side?
   If you missed this problem, review **Example 3.35**.

Solve Proportions

When two rational expressions are equal, the equation relating them is called a *proportion*.

Proportion

A **proportion** is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$.

The proportion is read "a is to b, as c is to d."

The equation $\frac{1}{2} = \frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2} = \frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Proportions are used in many applications to 'scale up' quantities. We'll start with a very simple example so you can see how proportions work. Even if you can figure out the answer to the example right away, make sure you also learn to solve it using proportions.

Suppose a school principal wants to have 1 teacher for 20 students. She could use proportions to find the number of teachers for 60 students. We let *x* be the number of teachers for 60 students and then set up the proportion:

$$\frac{1 \text{ teacher}}{20 \text{ students}} = \frac{x \text{ teachers}}{60 \text{ students}}$$

We are careful to match the units of the numerators and the units of the denominators—teachers in the numerators, students in the denominators.

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved equations in **Solve Rational Equations**. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

So let's finish solving the principal's problem now. We will omit writing the units until the last step.

	$\frac{1}{20} = \frac{x}{60}$
Multiply both sides by the LCD, 60.	$\frac{1}{20} \cdot 60 = \frac{x}{60} \cdot 60$
Simplify.	$3 = x$
	The principal needs 3 teachers for 60 students.

Now we'll do a few examples of solving numerical proportions without any units. Then we will solve applications using proportions.

EXAMPLE 8.71

Solve the proportion: $\frac{x}{63} = \frac{4}{7}$.

Solution

	$\frac{x}{63} = \frac{4}{7}$
To isolate x, multiply both sides by the LCD, 63.	$63\left(\frac{x}{63}\right) = 63\left(\frac{4}{7}\right)$
Simplify.	$x = \frac{9 \cdot \cancel{7} \cdot 4}{\cancel{7}}$
Divide the common factors.	$x = 36$
Check. To check our answer, we substitute into the original proportion.	
	$\frac{x}{63} = \frac{4}{7}$
Substitute $x = 36$.	$\frac{36}{63} \stackrel{?}{=} \frac{4}{7}$
Show common factors.	$\frac{4 \cdot 9}{7 \cdot 9} \stackrel{?}{=} \frac{4}{7}$
Simplify.	$\frac{4}{7} = \frac{4}{7}$ ✓

TRY IT : : 8.141 Solve the proportion: $\frac{n}{84} = \frac{11}{12}$.

TRY IT : : 8.142 Solve the proportion: $\frac{y}{96} = \frac{13}{12}$.

When we work with proportions, we exclude values that would make either denominator zero, just like we do for all rational expressions. What value(s) should be excluded for the proportion in the next example?

EXAMPLE 8.72

Solve the proportion: $\frac{144}{a} = \frac{9}{4}$.

Solution

	$\frac{144}{a} = \frac{9}{4}$
Multiply both sides by the LCD.	$\frac{144}{a} \cdot 4a = \frac{9}{4} \cdot 4a$
Remove common factors on each side.	$4 \cdot 144 = a \cdot 9$
Simplify.	$576 = 9a$
Divide both sides by 9.	$\frac{576}{9} = \frac{9a}{9}$

Simplify.	$64 = a$
Check.	
	$\frac{144}{a} = \frac{9}{4}$
Substitute $a = 64$.	$\frac{144}{64} \stackrel{?}{=} \frac{9}{4}$
Show common factors.	$\frac{9 \cdot 16}{4 \cdot 16} \stackrel{?}{=} \frac{9}{4}$
Simplify.	$\frac{9}{4} = \frac{9}{4}$ ✓

TRY IT : : 8.143 Solve the proportion: $\frac{91}{b} = \frac{7}{5}$.

TRY IT : : 8.144 Solve the proportion: $\frac{39}{c} = \frac{13}{8}$.

EXAMPLE 8.73

Solve the proportion: $\frac{n}{n+14} = \frac{5}{7}$.

Solution

	$\frac{n}{n+14} = \frac{5}{7}$
Multiply both sides by the LCD.	$7(n+14)\left(\frac{n}{n+14}\right) = 7(n+14)\left(\frac{5}{7}\right)$
Remove common factors on each side.	$7n = 5(n+14)$
Simplify.	$7n = 5n + 70$
Solve for n.	$2n = 70$
	$n = 35$
Check.	
	$\frac{n}{n+14} = \frac{5}{7}$
Substitute $n = 35$.	$\frac{35}{35+14} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{35}{49} \stackrel{?}{=} \frac{5}{7}$
Show common factors.	$\frac{5 \cdot 7}{7 \cdot 7} \stackrel{?}{=} \frac{5}{7}$
Simplify.	$\frac{5}{7} = \frac{5}{7}$ ✓

TRY IT : : 8.145 Solve the proportion: $\frac{y}{y+55} = \frac{3}{8}$.

TRY IT : : 8.146 Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

EXAMPLE 8.74

Solve: $\frac{p+12}{9} = \frac{p-12}{6}$.

Solution

	$\frac{p+12}{9} = \frac{p-12}{6}$
Multiply both sides by the LCD, 18.	$18\left(\frac{p+12}{9}\right) = 18\left(\frac{p-12}{6}\right)$
Simplify.	$2(p+12) = 3(p-12)$
Distribute.	$2p + 24 = 3p - 36$
Solve for p.	$60 = p$
Check.	
	$\frac{p+12}{9} = \frac{p-12}{6}$
Substitute $p = 60$.	$\frac{60+12}{9} \stackrel{?}{=} \frac{60-12}{6}$
Simplify.	$\frac{72}{9} \stackrel{?}{=} \frac{48}{6}$
Divide.	$8 = 8$ ✓

TRY IT : : 8.147 Solve: $\frac{v+30}{8} = \frac{v+66}{12}$.

TRY IT : : 8.148 Solve: $\frac{2x+15}{9} = \frac{7x+3}{15}$.

To solve applications with proportions, we will follow our usual strategy for solving applications. But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match and the units in the denominators must match.

EXAMPLE 8.75

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many ml of acetaminophen will the doctor prescribe?
	Let $a = \text{ml}$ of acetaminophen.
Write a sentence that gives the information to find it.	If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?

Translate into a proportion–be careful of the units. $\frac{\text{ml}}{\text{pounds}} = \frac{\text{ml}}{\text{pounds}}$	$\frac{5}{25} = \frac{a}{80}$
Multiply both sides by the LCD, 400.	$400\left(\frac{5}{25}\right) = 400\left(\frac{a}{80}\right)$
Remove common factors on each side.	$\cancel{25} \cdot 16\left(\frac{5}{\cancel{25}}\right) = \cancel{80} \cdot 5\left(\frac{a}{\cancel{80}}\right)$
Simplify, but don't multiply on the left. Notice what the next step will be.	$16 \cdot 5 = 5a$
Solve for a.	$\frac{16 \cdot 5}{5} = \frac{5a}{5}$
Check.	$16 = a$
Is the answer reasonable?	
Yes, since 80 is about 3 times 25, the medicine should be about 3 times 5. So 16 ml makes sense.	
Substitute $a = 16$ in the original proportion. $\frac{5}{25} = \frac{a}{80}$ $\frac{5}{25} \stackrel{?}{=} \frac{16}{80}$ $\frac{1}{5} = \frac{1}{5}$ ✓	
Write a complete sentence.	The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

TRY IT : : 8.149

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

TRY IT : : 8.150

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

EXAMPLE 8.76

A 16-ounce iced caramel macchiato has 230 calories. How many calories are there in a 24-ounce iced caramel macchiato?

Solution

Identify what we are asked to find, and choose a variable to represent it.	How many calories are in a 24 ounce iced caramel macchiato?
	Let $c = \text{calories}$ in 24 ounces.
Write a sentence that gives the information to find it.	If there are 230 calories in 16 ounces, then how many calories are in 24 ounces?

Translate into a proportion–be careful of the units. $\frac{\text{calories}}{\text{ounce}} = \frac{\text{calories}}{\text{ounce}}$	$\frac{230}{16} = \frac{c}{24}$
Multiply both sides by the LCD, 48.	$48\left(\frac{230}{16}\right) = 48\left(\frac{c}{24}\right)$
Remove common factors on each side.	$\cancel{16} \cdot 3\left(\frac{230}{\cancel{16}}\right) = \cancel{24} \cdot 2\left(\frac{c}{\cancel{24}}\right)$
Simplify.	$690 = 2c$
Solve for c.	$\frac{690}{2} = \frac{2c}{2}$
	$345 = c$
Check.	
Is the answer reasonable?	
Yes, 345 calories for 24 ounces is more than 290 calories for 16 ounces, but not too much more.	
Substitute $c = 345$ in the original proportion. $\frac{230}{16} = \frac{c}{24}$ $\frac{230}{16} \overset{?}{=} \frac{345}{24}$ $\frac{115}{8} = \frac{115}{8}$ ✓	
Write a complete sentence.	There are 345 calories in a 24-ounce iced caramel macchiato.

TRY IT : : 8.151

At a fast-food restaurant, a 22-ounce chocolate shake has 850 calories. How many calories are in their 12-ounce chocolate shake? Round your answer to nearest whole number.

TRY IT : : 8.152

Yaneli loves Starburst candies, but wants to keep her snacks to 100 calories. If the candies have 160 calories for 8 pieces, how many pieces can she have in her snack?

EXAMPLE 8.77

Josiah went to Mexico for spring break and changed $325 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 12.54 Mexican pesos. How many Mexican pesos did he get for his trip?

Solution

What are you asked to find?	How many Mexican pesos did Josiah get?
Assign a variable.	Let $p =$ the number of Mexican pesos.

Write a sentence that gives the information to find it.	If \$1 US is equal to 12.54 Mexican pesos, then \$325 is how many pesos?
Translate into a proportion–be careful of the units.	
$\frac{\$}{\text{pesos}} = \frac{\$}{\text{pesos}}$	$\frac{1}{12.54} = \frac{325}{p}$
Multiply both sides by the LCD, $12.54p$.	$12.54p\left(\frac{1}{12.54}\right) = 12.54p\left(\frac{325}{p}\right)$
Remove common factors on each side.	$\cancel{12.54}p\left(\frac{1}{\cancel{12.54}}\right) = 12.54\cancel{p}\left(\frac{325}{\cancel{p}}\right)$
Simplify.	$p = 4075.5$
Check.	
Is the answer reasonable?	
Yes, \$100 would be 1,254 pesos. \$325 is a little more than 3 times this amount, so our answer of 4075.5 pesos makes sense.	
Substitute $p = 4075.5$ in the original proportion. Use a calculator. $\frac{1}{12.54} = \frac{325}{p}$ $\frac{1}{12.54} \stackrel{?}{=} \frac{325}{4075.5}$ $0.07874... = 0.07874...$ ✓	
Write a complete sentence.	Josiah got 4075.5 pesos for his spring break trip.

TRY IT : : 8.153

Yurianna is going to Europe and wants to change \$800 dollars into Euros. At the current exchange rate, \$1 US is equal to 0.738 Euro. How many Euros will she have for her trip?

TRY IT : : 8.154

Corey and Nicole are traveling to Japan and need to exchange \$600 into Japanese yen. If each dollar is 94.1 yen, how many yen will they get?

In the example above, we related the number of pesos to the number of dollars by using a proportion. We could say the number of pesos *is proportional to* the number of dollars. If two quantities are related by a proportion, we say that they are proportional.

Solve Similar Figure Applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with **similar figures**. If two figures have exactly the same shape, but different sizes, they are said to be *similar*. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides are in the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

For example, the two triangles in **Figure 8.2** are similar. Each side of ΔABC is 4 times the length of the corresponding side of ΔXYZ.

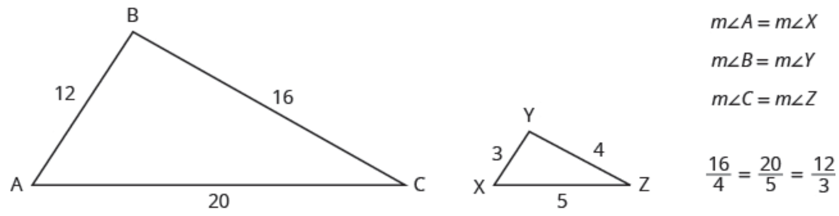
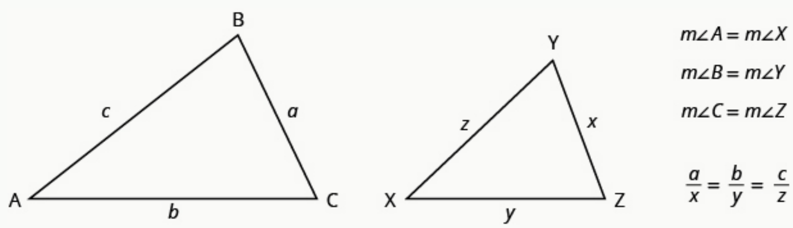


Figure 8.2

This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides are in the same ratio.



To solve applications with similar figures we will follow the Problem-Solving Strategy for Geometry Applications we used earlier.

HOW TO : : SOLVE GEOMETRY APPLICATIONS.

Step 1. **Read** the problem and make all the words and ideas are understood. Draw the figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

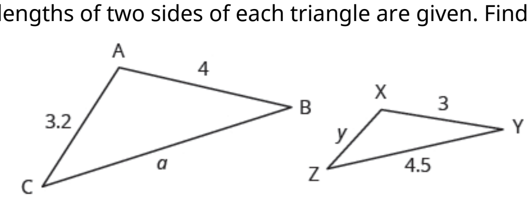
Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

EXAMPLE 8.78

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given. Find the lengths of the third sides.

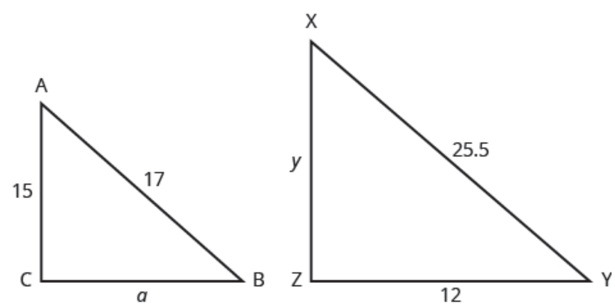


Solution

Step 1. Read the problem. Draw the figure and label it with the given information.	Figure is given.
Step 2. Identify what we are looking for.	the length of the sides of similar triangles
Step 3. Name the variables.	Let $a =$ length of the third side of ΔABC. $y =$ length of the third side of ΔXYZ
Step 4. Translate.	Since the triangles are similar, the corresponding sides are proportional.
We need to write an equation that compares the side we are looking for to a known ratio. Since the side *AB* = 4 corresponds to the side *XY* = 3 we know $\frac{AB}{XY} = \frac{4}{3}$. So we write equations with $\frac{AB}{XY}$ to find the sides we are looking for. Be careful to match up corresponding sides correctly.	$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$. To find *a*: sides of large triangle / sides of small triangle → $\frac{AB}{XY} = \frac{BC}{YZ}$. To find *y*: $\frac{AB}{XY} = \frac{AC}{XZ}$
Substitute.	$\frac{4}{3} = \frac{a}{4.5}$; $\frac{4}{3} = \frac{3.2}{y}$
Step 5. Solve the equation.	$3a = 4(4.5)$; $4y = 3(3.2)$
	$a = 6$; $y = 2.4$
Step 6. Check. $\frac{4}{3} \stackrel{?}{=} \frac{6}{4.5}$, $4(4.5) \stackrel{?}{=} 6(3)$, $18 = 18$ ✓ ; $\frac{4}{3} \stackrel{?}{=} \frac{3.2}{2.4}$, $4(2.4) \stackrel{?}{=} 3.2(3)$, $9.6 = 9.6$ ✓	
Step 7. Answer the question.	The third side of ΔABC is 6 and the third side of ΔXYZ is 2.4.

TRY IT : : 8.155

ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.



Find the length of side a.

TRY IT : : 8.156

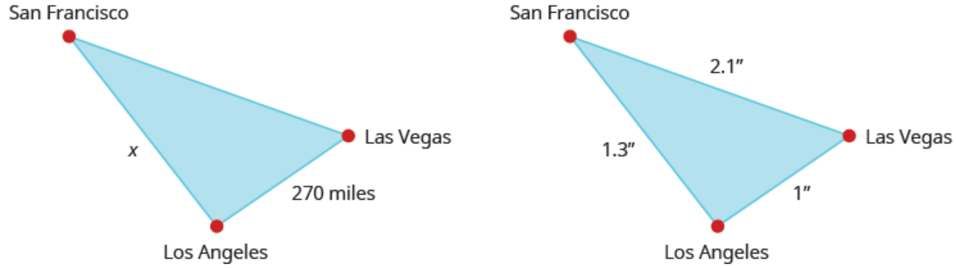
ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure.

Find the length of side y.

The next example shows how similar triangles are used with maps.

EXAMPLE 8.79

On a map, San Francisco, Las Vegas, and Los Angeles form a triangle whose sides are shown in the figure below. If the actual distance from Los Angeles to Las Vegas is 270 miles find the distance from Los Angeles to San Francisco.

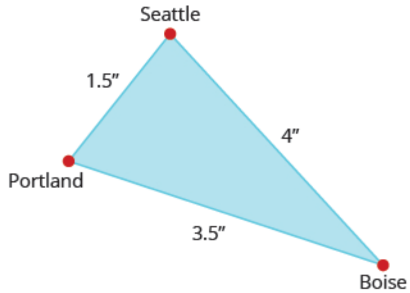


Solution

Read the problem. Draw the figures and label with the given information.	The figures are shown above.
Identify what we are looking for.	The actual distance from Los Angeles to San Francisco.
Name the variables.	Let $x =$ distance from Los Angeles to San Francisco.
Translate into an equation. Since the triangles are similar, the corresponding sides are proportional. We'll make the numerators "miles" and the denominators "inches."	$\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$
Solve the equation.	$1.3\left(\frac{x}{1.3}\right) = 1.3\left(\frac{270}{1}\right)$
	$x = 351$
Check.	
On the map, the distance from Los Angeles to San Francisco is more than the distance from Los Angeles to Las Vegas. Since 351 is more than 270 the answer makes sense.	
Check $x = 351$ in the original proportion. Use a calculator. $\frac{x \text{ miles}}{1.3 \text{ inches}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{351 \text{ miles}}{1.3 \text{ inches}} \stackrel{?}{=} \frac{270 \text{ miles}}{1 \text{ inch}}$ $\frac{270 \text{ miles}}{1 \text{ inch}} = \frac{270 \text{ miles}}{1 \text{ inch}}$ ✓	
Answer the question.	The distance from Los Angeles to San Francisco is 351 miles.

TRY IT : : 8.157

On the map, Seattle, Portland, and Boise form a triangle whose sides are shown in the figure below. If the actual distance from Seattle to Boise is 400 miles, find the distance from Seattle to Portland.



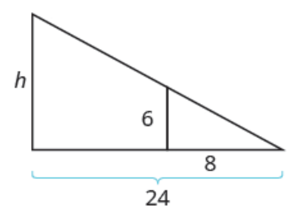
TRY IT : : 8.158 Using the map above, find the distance from Portland to Boise.

We can use similar figures to find heights that we cannot directly measure.

EXAMPLE 8.80

Tyler is 6 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a tree was 24 feet long. Find the height of the tree.

Solution

Read the problem and draw a figure.	
We are looking for *h*, the height of the tree.	
We will use similar triangles to write an equation.	
The small triangle is similar to the large triangle.	$\frac{h}{24}=\frac{6}{8}$
Solve the proportion.	$24\left(\frac{6}{8}\right)=24\left(\frac{h}{24}\right)$
Simplify.	$18=h$
Check.	
Tyler's height is less than his shadow's length so it makes sense that the tree's height is less than the length of its shadow.	

Check $h=18$ in the original proportion.

$$\frac{6}{8}=\frac{h}{24}$$

$$\frac{6}{8}\stackrel{?}{=}\frac{18}{24}$$

$$\frac{3}{4}=\frac{3}{4}\checkmark$$

TRY IT : : 8.159

A telephone pole casts a shadow that is 50 feet long. Nearby, an 8 foot tall traffic sign casts a shadow that is 10 feet long. How tall is the telephone pole?

TRY IT : : 8.160

A pine tree casts a shadow of 80 feet next to a 30-foot tall building which casts a 40 feet shadow. How tall is the pine tree?

8.7 EXERCISES

Practice Makes Perfect

Solve Proportions

In the following exercises, solve.

365. $\frac{x}{56} = \frac{7}{8}$

366. $\frac{n}{91} = \frac{8}{13}$

367. $\frac{49}{63} = \frac{z}{9}$

368. $\frac{56}{72} = \frac{y}{9}$

369. $\frac{5}{a} = \frac{65}{117}$

370. $\frac{4}{b} = \frac{64}{144}$

371. $\frac{98}{154} = \frac{-7}{p}$

372. $\frac{72}{156} = \frac{-6}{q}$

373. $\frac{a}{-8} = \frac{-42}{48}$

374. $\frac{b}{-7} = \frac{-30}{42}$

375. $\frac{2.7}{j} = \frac{0.9}{0.2}$

376. $\frac{2.8}{k} = \frac{2.1}{1.5}$

377. $\frac{a}{a+12} = \frac{4}{7}$

378. $\frac{b}{b-16} = \frac{11}{9}$

379. $\frac{c}{c-104} = -\frac{5}{8}$

380. $\frac{d}{d-48} = -\frac{13}{3}$

381. $\frac{m+90}{25} = \frac{m+30}{15}$

382. $\frac{n+10}{4} = \frac{40-n}{6}$

383. $\frac{2p+4}{8} = \frac{p+18}{6}$

384. $\frac{q-2}{2} = \frac{2q-7}{18}$

385. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

386. Brianna, who weighs 6 kg, just received her shots and needs a pain killer. The pain killer is prescribed for children at 15 milligrams (mg) for every 1 kilogram (kg) of the child's weight. How many milligrams will the doctor prescribe?

387. A veterinarian prescribed Sunny, a 65 pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?

388. Belle, a 13 pound cat, is suffering from joint pain. How much medicine should the veterinarian prescribe if the dosage is 1.8 mg per pound?

389. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

390. One 12 ounce can of soda has 150 calories. If Josiah drinks the big 32 ounce size from the local mini-mart, how many calories does he get?

391. A new 7 ounce lemon ice drink is advertised for having only 140 calories. How many ounces could Sally drink if she wanted to drink just 100 calories?

392. Reese loves to drink healthy green smoothies. A 16 ounce serving of smoothie has 170 calories. Reese drinks 24 ounces of these smoothies in one day. How many calories of smoothie is he consuming in one day?

393. Janice is traveling to Canada and will change \$250 US dollars into Canadian dollars. At the current exchange rate, \$1 US is equal to \$1.01 Canadian. How many Canadian dollars will she get for her trip?

394. Todd is traveling to Mexico and needs to exchange \$450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?

395. Steve changed \$600 into 480 Euros. How many Euros did he receive for each US dollar?

396. Martha changed \$350 US into 385 Australian dollars. How many Australian dollars did she receive for each US dollar?

397. When traveling to Great Britain, Bethany exchanged her \$900 into 570 British pounds. How many pounds did she receive for each American dollar?

398. A missionary commissioned to South Africa had to exchange his $500 for the South African Rand which is worth 12.63 for every dollar. How many Rand did he have after the exchange?

399. Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

400. Sarah drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?

401. Elizabeth is returning to the United States from Canada. She changes the remaining 300 Canadian dollars she has to $230.05 in American dollars. What was $1 worth in Canadian dollars?

402. Ben needs to convert $1000 to the Japanese Yen. One American dollar is worth 123.3 Yen. How much Yen will he have?

403. A golden retriever weighing 85 pounds has diarrhea. His medicine is prescribed as 1 teaspoon per 5 pounds. How much medicine should he be given?

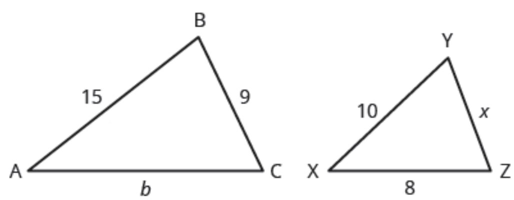
404. Five-year-old Lacy was stung by a bee. The dosage for the anti-itch liquid is 150 mg for her weight of 40 pounds. What is the dosage per pound?

405. Karen eats $\frac{1}{2}$ cup of oatmeal that counts for 2 points on her weight loss program. Her husband, Joe, can have 3 points of oatmeal for breakfast. How much oatmeal can he have?

406. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

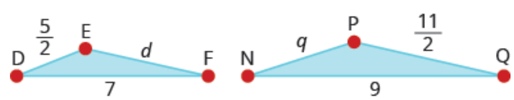
In the following exercises, ΔABC is similar to ΔXYZ. Find the length of the indicated side.



407. side *b*

408. side *x*

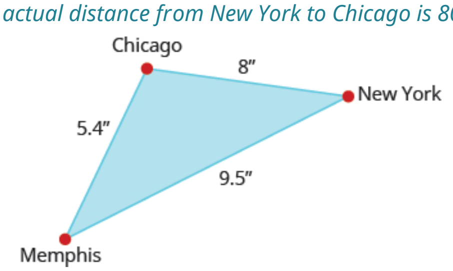
In the following exercises, ΔDEF is similar to ΔNPQ.



409. Find the length of side *d*.

410. Find the length of side *q*.

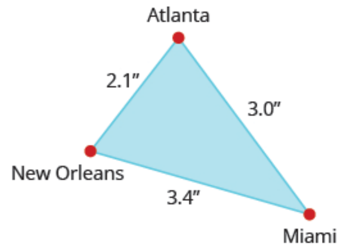
In the following two exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle whose sides are shown in the figure below. The actual distance from New York to Chicago is 800 miles.



411. Find the actual distance from New York to Memphis.

412. Find the actual distance from Chicago to Memphis.

In the following two exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle whose sides are shown in the figure below. The actual distance from Atlanta to New Orleans is 420 miles.



413. Find the actual distance from New Orleans to Miami.

414. Find the actual distance from Atlanta to Miami.

415. A 2 foot tall dog casts a 3 foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

416. Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?

417. The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven foot shadow. How long is the windmill's shadow?

418. The height of the Statue of Liberty is 305 feet. Nicole, who is standing next to the statue, casts a 6 foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Everyday Math

419. Heart Rate At the gym, Carol takes her pulse for 10 seconds and counts 19 beats.

ⓐ How many beats per minute is this?

ⓑ Has Carol met her target heart rate of 140 beats per minute?

420. Heart Rate Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.

ⓐ How many beats per minute is this?

ⓑ Has Kevin met his target heart rate?

421. Cost of a Road Trip Jesse's car gets 30 miles per gallon of gas.

ⓐ If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?

ⓑ If gas is $3.09 per gallon, what is the total cost of the gas for the trip?

422. Cost of a Road Trip Danny wants to drive to Phoenix to see his grandfather. Phoenix is 370 miles from Danny's home and his car gets 18.5 miles per gallon.

ⓐ How many gallons of gas will Danny need to get to and from Phoenix?

ⓑ If gas is $3.19 per gallon, what is the total cost for the gas to drive to see his grandfather?

423. Lawn Fertilizer Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

424. House Paint April wants to paint the exterior of her house. One gallon of paint covers about 350 square feet, and the exterior of the house measures approximately 2000 square feet. How many gallons of paint will she have to buy?

425. Cooking Natalia's pasta recipe calls for 2 pounds of pasta for 1 quart of sauce. How many pounds of pasta should Natalia cook if she has 2.5 quarts of sauce?

426. Heating Oil A 275 gallon oil tank costs $400 to fill. How much would it cost to fill a 180 gallon oil tank?

Writing Exercises

427. Marisol solves the proportion $\frac{144}{a}=\frac{9}{4}$ by 'cross multiplying', so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in **Example 8.72**.

428. Find a printed map and then write and solve an application problem similar to **Example 8.79**.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve proportions.			
solve similar figure applications.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.8 Solve Uniform Motion and Work Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve uniform motion applications
- Solve work applications

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. An express train and a local bus leave Chicago to travel to Champaign. The express bus can make the trip in 2 hours and the local bus takes 5 hours for the trip. The speed of the express bus is 42 miles per hour faster than the speed of the local bus. Find the speed of the local bus.
If you missed this problem, review **Example 3.48**.
2. Solve $\frac{1}{3}x + \frac{1}{4}x = \frac{5}{6}$.
If you missed this problem, review **Example 3.49**.
3. Solve: $18t^2 - 30 = -33t$.
If you missed this problem, review **Example 7.79**.

Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

The formula $D = rt$ assumes we know *r* and *t* and use them to find D. If we know D and *r* and need to find *t*, we would solve the equation for *t* and get the formula $t = \frac{D}{r}$.

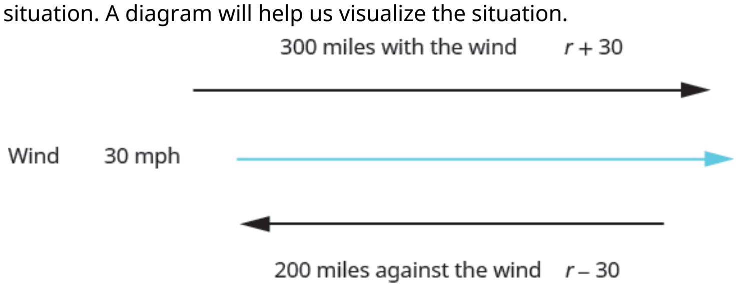
We have also explained how flying with or against a current affects the speed of a vehicle. We will revisit that idea in the next example.

EXAMPLE 8.81

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane.	Let $r =$ the speed of the airplane.
When the plane flies with the wind, the wind increases its speed and the rate is $r + 30$.	
When the plane flies against the wind, the wind decreases its speed and the rate is $r - 30$.	
Write in the rates. Write in the distances. Since $D = r \bullet t$, we solve for t and get $\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	(see table below)
We know the times are equal and so we write our equation.	$\frac{200}{r-30} = \frac{300}{r+30}$
We multiply both sides by the LCD. $200(r+30) = 300(r-30)$	$(r+30)(r-30)\left(\frac{200}{r-30}\right) = (r+30)(r-30)\left(\frac{300}{r+30}\right)$
Simplify.	$(r+30)(200) = (r-30)(300)$
	$200r + 6000 = 300r - 9000$
Solve.	$15000 = 100r$ $150 = r$
Check.	
Is 150 mph a reasonable speed for an airplane? Yes. If the plane is traveling 150 mph and the wind is 30 mph:	
Tailwind $150 + 30 = 180\text{mph}$ $\frac{300}{180} = \frac{5}{3}$ hours	
Headwind $150 - 30 = 120\text{mph}$ $\frac{200}{120} = \frac{5}{3}$ hours	
The times are equal, so it checks.	The plane was traveling 150 mph.

	Rate	• Time	= Distance
Headwind	$r-30$	$\frac{200}{r-30}$	200
Tailwind	$r+30$	$\frac{300}{r+30}$	300

TRY IT : : 8.161

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

TRY IT : : 8.162

Judy can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Judy's boat without a wind?

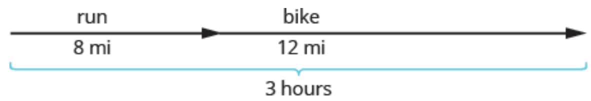
In the next example, we will know the total time resulting from travelling different distances at different speeds.

EXAMPLE 8.82

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Jazmine's running speed. — Let $r =$ Jazmine's running speed.

Her biking speed is 4 miles faster than her running speed. — $r + 4 =$ her biking speed

The distances are given, enter them into the chart.

Since $D = r \bullet t$, we solve for t and get $t = \frac{D}{r}$.

We divide the distance by the rate in each row, and place the expression in the time column.

	Rate •	Time =	Distance
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

Write a word sentence. — Her time plus the time biking is 3 hours.

Translate the sentence to get the equation.

$$\frac{8}{r} + \frac{24}{r+4} = 3$$

Solve.

$$\begin{aligned} r(r+4)\left(\frac{8}{r} + \frac{24}{r+4}\right) &= 3 \bullet r(r+4) \\ 8(r+4) + 24r &= 3r(r+4) \\ 8r + 32 + 24r &= 3r^2 + 12r \\ 32 + 32r &= 3r^2 + 12r \\ 0 &= 3r^2 - 20r - 32 \\ 0 &= (3r+4)(r-8) \end{aligned}$$

$$(3r+4) = 0 \quad (r-8) = 0$$

$$r = -\frac{4}{3} \quad r = 8$$

Check. ~~$r = -\frac{4}{3}$~~ $r = 8$

A negative speed does not make sense in this problem, so $r = 8$ is the solution.

Is 8 mph a reasonable running speed? Yes.

Run 8 mph $\quad \frac{8 \text{ miles}}{8 \text{ mph}} = 1 \text{ hour}$

Bike 12 mph $\quad \frac{24 \text{ miles}}{12 \text{ mph}} = 2 \text{ hours}$

Total 3 hours

Jazmine's running speed is 8 mph.

TRY IT : : 8.163

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

TRY IT : : 8.164

Tony drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

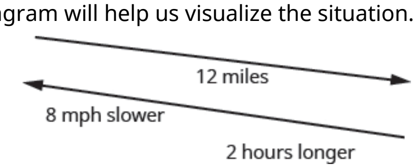
Once again, we will use the uniform motion formula solved for the variable *t*.

EXAMPLE 8.83

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed.	Let $r =$ Hamilton's downhill speed.
His uphill speed is 8 miles per hour slower. Enter the rates into the chart.	$h - 8 =$ Hamilton's uphill speed
The distance is the same in both directions, 12 miles. Since $D = r \bullet t$, we solve for *t* and get $t = \frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.	(see chart below)
Write a word sentence about the time.	He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.
Translate the sentence to get the equation. Solve.	(see below)
Check. Is 12 mph a reasonable speed for biking downhill? Yes.	

	Rate •	Time =	Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

$$\frac{12}{h-8} = \frac{12}{h} + 2$$

$$h(h-8)\left(\frac{12}{h-8}\right) = h(h-8)\left(\frac{12}{h} + 2\right)$$

$$12h = 12(h-8) + 2h(h-8)$$

$$12h = 12h - 96 + 2h^2 - 16h$$

$$0 = 2h^2 - 16h - 96$$

$$0 = 2\left(h^2 - 8h - 48\right)$$

$$0 = 2(h-12)(h+4)$$

$$h - 12 = 0 \quad h + 4 = 0$$

$$h = 12 \quad \cancel{h = -4}$$

Downhill	12 mph	$\frac{12 \text{ miles}}{12 \text{ mph}} = 1 \text{ hour}$	
Uphill	$12 - 8 = 4$ mph	$\frac{12 \text{ miles}}{4 \text{ mph}} = 3 \text{ hours}$	
			The uphill time is 2 hours more than the downhill time. Hamilton's downhill speed is 12 mph.

TRY IT : : 8.165

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

TRY IT : : 8.166

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 18 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Solve Work Applications

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

This is a typical 'work' application. There are three quantities involved here – the time it would take each of the two people to do the job alone and the time it would take for them to do the job together.

Let's get back to Pete and Alicia painting the room. We will let t be the number of hours it would take them to paint the room together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

	Rate	• Time	= Distance
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take them if they worked together, we solve for t.

$$\underbrace{\text{Pete's part}}_{\frac{1}{10}} + \underbrace{\text{Alicia's part}}_{\frac{1}{8}} = \underbrace{\text{part of total}}_{\frac{1}{t}}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

$$\frac{1}{10} + \frac{1}{8} = \frac{1}{t}$$

Multiply by the LCD, $40t$.	$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$
Distribute.	$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$
Simplify and solve.	$4t+5t=40$
	$9t=40$
	$t=\frac{40}{9}$
We'll write as a mixed number so that we can convert it to hours and minutes.	$t=4\frac{4}{9}$ hours
Remember, 1 hour = 60 minutes.	$t=4$ hours $+\frac{4}{9}(60$ minutes$)$
Multiply, and then round to the nearest minute.	$t=4$ hours $+27$ minutes
	It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

Keep in mind, it should take less time for two people to complete a job working together than for either person to do it alone.

EXAMPLE 8.84

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

Solution

This is a work problem. A chart will help us organize the information.

Let $t =$ the number of hours needed to complete the job together.	
Enter the hours per job for Press #1, Press #2 and when they work together. If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed. Similarly find the part of the job completed/hours for Press #2 and when they both work together.	(see table below)
Write a word sentence.	
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.	
Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Solve.	$\frac{1}{6} + \frac{1}{12} = \frac{1}{t}$
Multiply by the LCD, $12t$.	$12t\left(\frac{1}{6} + \frac{1}{12}\right) = 12t\left(\frac{1}{t}\right)$
Simplify.	$2t + t = 12$
	$3t = 12$
	$t = 4$
	When both presses are running it takes 4 hours to do the job.

	Number of hours to complete the job	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

TRY IT : : 8.167

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

TRY IT : : 8.168

Carrie can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

EXAMPLE 8.85

Corey can shovel all the snow from the sidewalk and driveway in 4 hours. If he and his twin Casey work together, they can finish shoveling the snow in 2 hours. How many hours would it take Casey to do the job by himself?

Solution

This is a work application. A chart will help us organize the information.	
We are looking for how many hours it would take Casey to complete the job by himself.	
Let $t =$ the number of hours needed for Casey to complete.	
Enter the hours per job for Corey, Casey, and when they work together. If Corey takes 4 hours, then in 1 hour $\frac{1}{4}$ of the job is completed. Similarly find the part of the job completed/hours for Casey and when they both work together.	(see chart below)
Write a word sentence.	
The part completed by Corey plus the part completed by Casey equals the amount completed together.	
Translate to an equation:	Corey + Casey = Together $\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Solve.	$\frac{1}{4} + \frac{1}{t} = \frac{1}{2}$
Multiply by the LCD, $4t$.	$4t\left(\frac{1}{4} + \frac{1}{t}\right) = 4t\left(\frac{1}{2}\right)$
Simplify.	$t + 4 = 2t$
	$4 = t$
	It would take Casey 4 hours to do the job alone.

	Number of hours needed to complete the job	Part of job completed/hour
Corey	4	$\frac{1}{4}$
Casey	t	$\frac{1}{t}$
Together	2	$\frac{1}{2}$

TRY IT : : 8.169

Two hoses can fill a swimming pool in 10 hours. It would take one hose 26 hours to fill the pool by itself. How long would it take for the other hose, working alone, to fill the pool?

TRY IT : : 8.170

Cara and Cindy, working together, can rake the yard in 4 hours. Working alone, it takes Cindy 6 hours to rake the yard. How long would it take Cara to rake the yard alone?

8.8 EXERCISES

Practice Makes Perfect

Solve Uniform Motion Applications

In the following exercises, solve uniform motion applications

429. Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35 mph headwind in the same amount of time it can travel 702 miles with a 35 mph tailwind. Find the speed of the helicopter.

430. A private jet can fly 1210 miles against a 25 mph headwind in the same amount of time it can fly 1694 miles with a 25 mph tailwind. Find the speed of the jet.

431. A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

432. Darrin can skateboard 2 miles against a 4 mph wind in the same amount of time he skateboards 6 miles with a 4 mph wind. Find the speed Darrin skateboards with no wind.

433. Jane spent 2 hours exploring a mountain with a dirt bike. When she rode the 40 miles uphill, she went 5 mph slower than when she reached the peak and rode for 12 miles along the summit. What was her rate along the summit?

434. Jill wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?

435. Bill wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

436. Nancy took a 3 hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?

437. Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, l, what was his uphill rate?

438. Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?

439. Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

440. Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel?

441. John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

442. Jim's speedboat can travel 20 miles upstream against a 3 mph current in the same amount of time it travels 22 miles downstream with a 3 mph current speed. Find the speed of the Jim's boat.

443. Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

444. Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?

445. When driving the 9 hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

446. Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?

Solve Work Applications

In the following exercises, solve work applications.

447. Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

448. It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?

449. Mary can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

450. Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?

451. Leeson can proofread a newspaper copy in 4 hours. If Ryan helps, they can do the job in 3 hours. How long would it take for Ryan to do his job alone?

452. Paul can clean a classroom floor in 3 hours. When his assistant helps him, the job takes 2 hours. How long would it take the assistant to do it alone?

453. Josephine can correct her students' test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

454. Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take the Levi's dad to wash the car by himself?

455. Jackson can remove the shingles off of a house in 7 hours, while Martin can remove the shingles in 5 hours. How long will it take them to remove the shingles if they work together?

456. At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

457. Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

458. It takes Tina 3 hours to frost her holiday cookies, but if Candy helps her it takes 2 hours. How long would it take Candy to frost the holiday cookies by herself?

Everyday Math

459. Dana enjoys taking her dog for a walk, but sometimes her dog gets away and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

460. Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Writing Exercises

461. In **Example 8.83**, the solution $h = -4$ is crossed out. Explain why.

462. Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3} + \frac{1}{4} = \frac{1}{t}$ can be used to find *t*, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve uniform motion applications.			
solve work applications.			

ⓑ *On a scale of 1–10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.9 Use Direct and Inverse Variation

Learning Objectives

By the end of this section, you will be able to:

- Solve direct variation problems
- Solve inverse variation problems

Be Prepared!

Before you get started, take this readiness quiz.

If you miss a problem, go back to the section listed and review the material.

1. Find the multiplicative inverse of -8.
   If you missed this problem, review **Example 1.126**.
2. Solve for n: $45 = 20n$.
   If you missed this problem, review **Example 2.13**.
3. Evaluate $5x^2$ when $x = 10$.
   If you missed this problem, review **Example 1.20**.

When two quantities are related by a proportion, we say they are *proportional* to each other. Another way to express this relation is to talk about the *variation* of the two quantities. We will discuss direct variation and inverse variation in this section.

Solve Direct Variation Problems

Lindsay gets paid \$15 per hour at her job. If we let *s* be her salary and *h* be the number of hours she has worked, we could model this situation with the equation

$$s = 15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary *varies directly* with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables *x* and *y*, *y* varies directly with *x* if

$$y = kx, \quad \text{where } k \neq 0$$

The constant *k* is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates *x* and *y*. Then we can use that equation to find values of *y* for other values of *x*.

EXAMPLE 8.86 HOW TO SOLVE DIRECT VARIATION PROBLEMS

If y varies directly with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution

Step 1. Write the formula for direct variation.	The direct variation formula is $y = kx$.	$y = kx$
Step 2. Substitute the given values for the variables.	We are given $y = 20$, $x = 8$.	$20 = k \cdot 8$
Step 3. Solve for the constant of variation.	Divide both sides of the equation by 8, then simplify.	$\frac{20}{8} = k$ $k = 2.5$
Step 4. Write the equation that relates *x* and *y*.	Rewrite the general equation with the value we found for *k*.	$y = 2.5x$

TRY IT : : 8.171 If y varies directly as x and $y = 3,$ when $x = 10.$ find the equation that relates *x* and *y*.

TRY IT : : 8.172 If y varies directly as x and $y = 12$ when $x = 4$ find the equation that relates *x* and *y*.

We'll list the steps below.

HOW TO : : SOLVE DIRECT VARIATION PROBLEMS.

Step 1. Write the formula for direct variation.
Step 2. Substitute the given values for the variables.
Step 3. Solve for the constant of variation.
Step 4. Write the equation that relates x and y.

Now we'll solve a few applications of direct variation.

EXAMPLE 8.87

When Raoul runs on the treadmill at the gym, the number of calories, *c*, he burns varies directly with the number of minutes, *m*, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

ⓐ Write the equation that relates *c* and *m*.

ⓑ How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution

ⓐ

	The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.
Write the formula for direct variation.	$y = kx$
We will use c in place of y and m in place of x.	$c = km$
Substitute the given values for the variables.	$315 = k \cdot 18$
Solve for the constant of variation.	$\frac{315}{18} = \frac{k \cdot 18}{18}$
	$17.5 = k$
Write the equation that relates c and m.	$c = km$
Substitute in the constant of variation.	$c = 17.5m$

ⓑ

	Find c when $m = 25$.
Write the equation that relates c and m.	$c = 17.5m$
Substitute the given value for m.	$c = 17.5(25)$
Simplify.	$c = 437.5$

	Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

TRY IT : : 8.173

The number of calories, *c*, burned varies directly with the amount of time, *t,* spent exercising. Arnold burned 312 calories in 65 minutes exercising.

ⓐ Write the equation that relates *c* and *t*.

ⓑ How many calories would he burn if he exercises for 90 minutes?

TRY IT : : 8.174

The distance a moving body travels, *d*, varies directly with time, *t*, it moves. A train travels 100 miles in 2 hours

ⓐ Write the equation that relates *d* and *t*.

ⓑ How many miles would it travel in 5 hours?

In the previous example, the variables *c* and *m* were named in the problem. Usually that is not the case. We will have to name the variables in the next example as part of the solution, just like we do in most applied problems.

EXAMPLE 8.88

The number of gallons of gas Eunice's car uses varies directly with the number of miles she drives. Last week she drove 469.8 miles and used 14.5 gallons of gas.

ⓐ Write the equation that relates the number of gallons of gas used to the number of miles driven.

ⓑ How many gallons of gas would Eunice's car use if she drove 1000 miles?

Solution

	The number of gallons of gas varies directly with the number of miles driven.
First we will name the variables.	Let $g =$ number of gallons of gas. $m =$ number of miles driven
Write the formula for direct variation.	$y = kx$
We will use g in place of y and m in place of x.	$g = km$
Substitute the given values for the variables.	$g = 14.5$ when $m = 469.8$
	$14.5 = k(469.8)$
Solve for the constant of variation.	$\frac{14.5}{469.8} = \frac{k(469.8)}{469.8}$
We will round to the nearest thousandth.	$0.031 = k$
Write the equation that relates g and m.	$g = km$
Substitute in the constant of variation.	$g = 0.031m$

ⓑ

	Find g when $m = 1000$.
Write the equation that relates g and m.	$g = 0.031m$
Substitute the given value for m.	$g = 0.031(1000)$
Simplify.	$g = 31$
	Eunice's car would use 31 gallons of gas if she drove it 1,000 miles.

Notice that in this example, the units on the constant of variation are gallons/mile. In everyday life, we usually talk about miles/gallon.

TRY IT : : 8.175

The distance that Brad travels varies directly with the time spent traveling. Brad travelled 660 miles in 12 hours,

ⓐ Write the equation that relates the number of miles travelled to the time.

ⓑ How many miles could Brad travel in 4 hours?

TRY IT : : 8.176

The weight of a liquid varies directly as its volume. A liquid that weighs 24 pounds has a volume of 4 gallons.

ⓐ Write the equation that relates the weight to the volume.

ⓑ If a liquid has volume 13 gallons, what is its weight?

In some situations, one variable varies directly with the square of the other variable. When that happens, the equation of direct variation is $y = k x^2$. We solve these applications just as we did the previous ones, by substituting the given values into the equation to solve for *k*.

EXAMPLE 8.89

The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 4" will support a maximum load of 75 pounds.

ⓐ Write the equation that relates the maximum load to the cross-section.

ⓑ What is the maximum load that can be supported by a beam with diagonal 8"?

Solution

	The maximum load varies directly with the square of the diagonal of the cross-section.
Name the variables.	Let $L =$ maximum load. $c =$ the diagonal of the cross-section
Write the formula for direct variation, where y varies directly with the square of x.	$y = kx^2$
We will use L in place of y and c in place of x.	$L = kc^2$
Substitute the given values for the variables.	$L = 75$ when $c = 4$
	$75 = k \cdot 4^2$

Solve for the constant of variation.	$\frac{75}{16} = \frac{k \cdot 16}{16}$
	$4.6875 = k$
Write the equation that relates L and c.	$L = kc^2$
Substitute in the constant of variation.	$L = 4.6875c^2$

ⓑ

	Find L when $c = 8$.
Write the equation that relates L and c.	$L = 4.6875c^2$
Substitute the given value for c.	$L = 4.6875(8)^2$
Simplify.	$L = 300$
	A beam with diagonal 8” could support a maximum load of 300 pounds.

TRY IT : : 8.177

The distance an object falls is directly proportional to the square of the time it falls. A ball falls 144 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.
ⓑ How far will an object fall in 4 seconds?

TRY IT : : 8.178

The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.

ⓑ What is the area of a pizza with a radius of 9 inches?

Solve Inverse Variation Problems

Many applications involve two variable that *vary inversely*. As one variable increases, the other decreases. The equation that relates them is $y = \frac{k}{x}$.

Inverse Variation

For any two variables *x* and *y*, *y* varies inversely with *x* if

$$y = \frac{k}{x}, \text{ where } k \neq 0$$

The constant *k* is called the constant of variation.

The word ‘inverse’ in inverse variation refers to the multiplicative inverse. The multiplicative inverse of *x* is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change ‘direct’ to ‘inverse’.

HOW TO : : SOLVE INVERSE VARIATION PROBLEMS.

Step 1. Write the formula for inverse variation.
Step 2. Substitute the given values for the variables.
Step 3. Solve for the constant of variation.
Step 4. Write the equation that relates x and y.

EXAMPLE 8.90

If *y* varies inversely with x and $y = 20$ when $x = 8$, find the equation that relates *x* and *y*.

Solution

Write the formula for inverse variation.	$y = \frac{k}{x}$
Substitute the given values for the variables.	$y = 20$ when $x = 8$
	$20 = \frac{k}{8}$
Solve for the constant of variation.	$8(20) = 8\left(\frac{k}{8}\right)$
	$160 = k$
Write the equation that relates x and y.	$y = \frac{k}{x}$
Substitute in the constant of variation.	$y = \frac{160}{x}$

TRY IT : : 8.179

If p varies inversely with q and $p = 30$ when $q = 12$ find the equation that relates p and q.

TRY IT : : 8.180

If y varies inversely with x and $y = 8$ when $x = 2$ find the equation that relates x and y.

EXAMPLE 8.91

The fuel consumption (mpg) of a car varies inversely with its weight. A car that weighs 3100 pounds gets 26 mpg on the highway.

ⓐ Write the equation of variation.

ⓑ What would be the fuel consumption of a car that weighs 4030 pounds?

Solution

ⓐ

	The fuel consumption varies inversely with the weight.

First we will name the variables.	Let $f =$ fuel consumption. $w =$ weight
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and w in place of x.	$f = \frac{k}{w}$
Substitute the given values for the variables.	$f = 26$ when $w = 3100$
	$26 = \frac{k}{3100}$
Solve for the constant of variation.	$3100(26) = 3100\left(\frac{k}{3100}\right)$
	$80{,}600 = k$
Write the equation that relates f and w.	$f = \frac{k}{w}$
Substitute in the constant of variation.	$f = \frac{80{,}600}{w}$

ⓑ

	Find f when $w = 4030$.
Write the equation that relates f and w.	$f = \frac{80{,}600}{w}$
Substitute the given value for w.	$f = \frac{80{,}600}{4030}$
Simplify.	$f = 20$
	A car that weighs 4030 pounds would have fuel consumption of 20 mpg.

TRY IT : : 8.181 A car's value varies inversely with its age. Elena bought a two-year-old car for $20,000.

ⓐ Write the equation of variation.
ⓑ What will be the value of Elena's car when it is 5 years old?

TRY IT : : 8.182

The time required to empty a pool varies inversely as the rate of pumping. It took Lucy 2.5 hours to empty her pool using a pump that was rated at 400 gpm (gallons per minute).

ⓐ Write the equation of variation.

ⓑ How long will it take her to empty the pool using a pump rated at 500 gpm?

EXAMPLE 8.92

The frequency of a guitar string varies inversely with its length. A 26" long string has a frequency of 440 vibrations per second.

ⓐ Write the equation of variation.

ⓑ How many vibrations per second will there be if the string's length is reduced to 20" by putting a finger on a fret?

Solution

ⓐ

	The frequency varies inversely with the length.
Name the variables.	Let $f =$ frequency. $L =$ length
Write the formula for inverse variation.	$y = \frac{k}{x}$
We will use f in place of y and L in place of x.	$f = \frac{k}{L}$
Substitute the given values for the variables.	$f = 440$ when $L = 26$
	$440 = \frac{k}{26}$
Solve for the constant of variation.	$26(440) = 26\left(\frac{k}{26}\right)$
	$11{,}440 = k$
Write the equation that relates f and L.	$f = \frac{k}{L}$
Substitute in the constant of variation.	$f = \frac{11{,}440}{L}$

ⓑ

	Find f when $L = 20$.
Write the equation that relates f and L.	$f = \frac{11{,}440}{L}$
Substitute the given value for L.	$f = \frac{11{,}440}{20}$
Simplify.	$f = 572$
	A 20” guitar string has frequency 572 vibrations per second.

TRY IT : : 8.183

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees.

ⓐ Write the equation of variation.

ⓑ How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

TRY IT : : 8.184

The force needed to break a board varies inversely with its length. Richard uses 24 pounds of pressure to break a 2-foot long board.

ⓐ Write the equation of variation.

ⓑ How many pounds of pressure is needed to break a 5-foot long board?

8.9 EXERCISES

Practice Makes Perfect

Solve Direct Variation Problems

In the following exercises, solve.

463. If y varies directly as x and $y = 14,$ when $x = 3$, find the equation that relates x and y.

464. If p varies directly as q and $p = 5,$ when $q = 2$, find the equation that relates p and q.

465. If v varies directly as w and $v = 24,$ when $w = 8$, find the equation that relates v and w.

466. If a varies directly as b and $a = 16,$ when $b = 4$, find the equation that relates a and b.

467. If p varies directly as q and $p = 9.6,$ when $q = 3$, find the equation that relates p and q.

468. If y varies directly as x and $y = 12.4,$ when $x = 4,$ find the equation that relates x and y

469. If a varies directly as b and $a = 6,$ when $b = \frac{1}{3}$, find the equation that relates a and b.

470. If v varies directly as w and $v = 8,$ when $w = \frac{1}{2}$, find the equation that relates v and w.

471. The amount of money Sally earns, *P*, varies directly with the number, *n*, of necklaces she sells. When Sally sells 15 necklaces she earns $150.

ⓐ Write the equation that relates *P* and *n*.

ⓑ How much money would she earn if she sold 4 necklaces?

472. The price, *P*, that Eric pays for gas varies directly with the number of gallons, *g*, he buys. It costs him $50 to buy 20 gallons of gas.

ⓐ Write the equation that relates *P* and *g*.

ⓑ How much would 33 gallons cost Eric?

473. Terri needs to make some pies for a fundraiser. The number of apples, *a*, varies directly with number of pies, *p*. It takes nine apples to make two pies.

ⓐ Write the equation that relates *a* and *p*.

ⓑ How many apples would Terri need for six pies?

474. Joseph is traveling on a road trip. The distance, *d*, he travels before stopping for lunch varies directly with the speed, *v*, he travels. He can travel 120 miles at a speed of 60 mph.

ⓐ Write the equation that relates *d* and *v*.

ⓑ How far would he travel before stopping for lunch at a rate of 65 mph?

475. The price of gas that Jesse purchased varies directly to how many gallons he purchased. He purchased 10 gallons of gas for $39.80.

ⓐ Write the equation that relates the price to the number of gallons.

ⓑ How much will it cost Jesse for 15 gallons of gas?

476. The distance that Sarah travels varies directly to how long she drives. She travels 440 miles in 8 hours.

ⓐ Write the equation that relates the distance to the number of hours.

ⓑ How far can Sally travel in 6 hours?

477. The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.

ⓐ Write the equation that relates the mass to the volume.

ⓑ What is the volume of this liquid if its mass is 128 kilograms?

478. The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10 pound watermelon on a hanging scale, the spring stretched 5 inches.

ⓐ Write the equation that relates the length of the spring to the weight.

ⓑ What weight of watermelon would stretch the spring 6 inches?

479. The distance an object falls varies directly to the square of the time it falls. A ball falls 45 feet in 3 seconds.

ⓐ Write the equation that relates the distance to the time.

ⓑ How far will the ball fall in 7 seconds?

480. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.

ⓐ Write the equation that relates the load to the diagonal of the cross-section.

ⓑ What load will a beam with a 10 inch diagonal support?

481. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.

ⓐ Write the equation that relates the area to the radius.

ⓑ What is the area of a personal pizza with a radius 4 inches?

482. The distance an object falls varies directly to the square of the time it falls. A ball falls 72 feet in 3 seconds,

ⓐ Write the equation that relates the distance to the time.

ⓑ How far will the ball have fallen in 8 seconds?

Solve Inverse Variation Problems

In the following exercises, solve.

483. If y varies inversely with x and $y=5$ when $x=4$ find the equation that relates x and y.

484. If p varies inversely with q and $p=2$ when $q=1$ find the equation that relates p and q.

485. If v varies inversely with w and $v=6$ when $w=\frac{1}{2}$ find the equation that relates v and w.

486. If a varies inversely with b and $a=12$ when $b=\frac{1}{3}$ find the equation that relates a and b.

Write an inverse variation equation to solve the following problems.

487. The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds and gets 33 mpg on the highway.

ⓐ Write the equation that relates the mpg to the car's weight.

ⓑ What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

488. A car's value varies inversely with its age. Jackie bought a 10 year old car for $2,400.

ⓐ Write the equation that relates the car's value to its age.

ⓑ What will be the value of Jackie's car when it is 15 years old ?

489. The time required to empty a tank varies inversely as the rate of pumping. It took Janet 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute),

ⓐ Write the equation that relates the number of hours to the pump rate.

ⓑ How long would it take Janet to pump her basement if she used a pump rated at 400 gpm?

490. The volume of a gas in a container varies inversely as pressure on the gas. A container of helium has a volume of 370 cubic inches under a pressure of 15 psi.

ⓐ Write the equation that relates the volume to the pressure.

ⓑ What would be the volume of this gas if the pressure was increased to 20 psi?

491. On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.

ⓐ Write the equation that relates the string length to its frequency.

ⓑ What is the frequency of a 10-inch string?

492. Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had 4 cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

ⓐ Write the equation that relates the number of cavities to the time spent brushing.

ⓑ How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

493. The number of tickets for a sports fundraiser varies inversely to the price of each ticket. Brianna can buy 25 tickets at $5each.

ⓐ Write the equation that relates the number of tickets to the price of each ticket.

ⓑ How many tickets could Brianna buy if the price of each ticket was $2.50?

494. Boyle's Law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.

ⓐ Write the equation that relates pressure to volume.

ⓑ If the pressure increases to 330 psi, how much air can Braydon's tank hold?

Mixed Practice

495. If y varies directly as x and $y = 5,\ \text{when}\ x = 3.$, find the equation that relates $x \text{ and } y$.

496. If v varies directly as w and $v = 21,\ \text{when}\ w = 8.$ find the equation that relates $v \text{ and } w.$

497. If p varies inversely with q and $p = 5$ when $q = 6$, find the equation that relates p and $q.$

498. If y varies inversely with x and $y = 11$ when $x = 3$ find the equation that relates x and $y.$

499. If p varies directly as q and $p = 10,\ \text{when}\ q = 2.$ find the equation that relates $p \text{ and } q$.

500. If v varies inversely with w and $v = 18$ when $w = \frac{1}{3}$ find the equation that relates v and $w.$

501. The force needed to break a board varies inversely with its length. If Tom uses 20 pounds of pressure to break a 1.5-foot long board, how many pounds of pressure would he need to use to break a 6 foot long board?

502. The number of hours it takes for ice to melt varies inversely with the air temperature. A block of ice melts in 2.5 hours when the temperature is 54 degrees. How long would it take for the same block of ice to melt if the temperature was 45 degrees?

503. The length a spring stretches varies directly with a weight placed at the end of the spring. When Meredith placed a 6-pound cantaloupe on a hanging scale, the spring stretched 2 inches. How far would the spring stretch if the cantaloupe weighed 9 pounds?

504. The amount that June gets paid varies directly the number of hours she works. When she worked 15 hours, she got paid $111. How much will she be paid for working 18 hours?

505. The fuel consumption (mpg) of a car varies inversely with its weight. A Ford Focus weighs 3000 pounds and gets 28.7 mpg on the highway. What would the fuel consumption be for a Ford Expedition that weighs 5,500 pounds? Round to the nearest tenth.

506. The volume of a gas in a container varies inversely as the pressure on the gas. If a container of argon has a volume of 336 cubic inches under a pressure of 2,500 psi, what will be its volume if the pressure is decreased to 2,000 psi?

507. The distance an object falls varies directly to the square of the time it falls. If an object falls 52.8 feet in 4 seconds, how far will it fall in 9 seconds?

508. The area of the face of a Ferris wheel varies directly with the square of its radius. If the area of one face of a Ferris wheel with diameter 150 feet is 70,650 square feet, what is the area of one face of a Ferris wheel with diameter of 16 feet?

Everyday Math

509. Ride Service It costs $35 for a ride from the city center to the airport, 14 miles away.

ⓐ Write the equation that relates the cost, *c*, with the number of miles, *m*.

ⓑ What would it cost to travel 22 miles with this service?

510. Road Trip The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.

ⓐ Write the equation that relates the number of hours, *h*, with the speed, *s.*

ⓑ How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

511. In your own words, explain the difference between direct variation and inverse variation.

512. Make up an example from your life experience of inverse variation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve direct variation problems.			
solve inverse variation problems.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next chapter? Why or why not?*

CHAPTER 8 REVIEW

KEY TERMS

complex rational expression A complex rational expression is a rational expression in which the numerator or denominator contains a rational expression.

extraneous solution to a rational equation An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

proportion A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b \neq 0,\ d \neq 0.$ The proportion is read " a is to b, as c is to $d.$"

rational equation A rational equation is two rational expressions connected by an equal sign.

rational expression A rational expression is an expression of the form $\frac{p}{q}$, where *p* and *q* are polynomials and $q \neq 0.$

similar figures Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides are in the same ratio.

KEY CONCEPTS

8.1 Simplify Rational Expressions

- **Determine the Values for Which a Rational Expression is Undefined**
  Step 1. Set the denominator equal to zero.
  Step 2. Solve the equation, if possible.
- **Simplified Rational Expression**
  - A rational expression is considered simplified if there are no common factors in its numerator and denominator.
- **Simplify a Rational Expression**
  Step 1. Factor the numerator and denominator completely.
  Step 2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  - The opposite of $a-b$ is $b-a$.
    $\frac{a-b}{b-a}=-1 \qquad a \neq 0,\ b \neq 0,\ \text{a} \neq \text{b}$

8.2 Multiply and Divide Rational Expressions

- **Multiplication of Rational Expressions**
  - If $p,\ q,\ r,\ s$ are polynomials where $q \neq 0,\ s \neq 0$, then $\frac{p}{q} \cdot \frac{r}{s}=\frac{pr}{qs}$.
  - To multiply rational expressions, multiply the numerators and multiply the denominators
- **Multiply a Rational Expression**
  Step 1. Factor each numerator and denominator completely.
  Step 2. Multiply the numerators and denominators.
  Step 3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  - If $p,\ q,\ r,\ s$ are polynomials where $q \neq 0,\ r \neq 0,\ s \neq 0$, then $\frac{p}{q} \div \frac{r}{s}=\frac{p}{q} \cdot \frac{s}{r}$.
  - To divide rational expressions multiply the first fraction by the reciprocal of the second.
- **Divide Rational Expressions**
  Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  Step 2. Factor the numerators and denominators completely.
  Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

8.3 Add and Subtract Rational Expressions with a Common Denominator

- **Rational Expression Addition**
  - If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}$$

  - To add rational expressions with a common denominator, add the numerators and place the sum over the common denominator.
- **Rational Expression Subtraction**
  - If $p, q,$ and r are polynomials where $r \neq 0$, then

$$\frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$

  - To subtract rational expressions, subtract the numerators and place the difference over the common denominator.

8.4 Add and Subtract Rational Expressions with Unlike Denominators

- **Find the Least Common Denominator of Rational Expressions**

Step 1. Factor each expression completely.

Step 2. List the factors of each expression. Match factors vertically when possible.

Step 3. Bring down the columns.

Step 4. Multiply the factors.

- **Add or Subtract Rational Expressions**

Step 1. Determine if the expressions have a common denominator.
Yes – go to step 2.
No – Rewrite each rational expression with the LCD.

  - Find the LCD.
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

8.5 Simplify Complex Rational Expressions

- **To Simplify a Rational Expression by Writing it as Division**

Step 1. Simplify the numerator and denominator.

Step 2. Rewrite the complex rational expression as a division problem.

Step 3. Divide the expressions.

- **To Simplify a Complex Rational Expression by Using the LCD**

Step 1. Find the LCD of all fractions in the complex rational expression.

Step 2. Multiply the numerator and denominator by the LCD.

Step 3. Simplify the expression.

8.6 Solve Rational Equations

- **Strategy to Solve Equations with Rational Expressions**

Step 1. Note any value of the variable that would make any denominator zero.

Step 2. Find the least common denominator of *all* denominators in the equation.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Step 4. Solve the resulting equation.

Step 5. Check.

- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

8.7 Solve Proportion and Similar Figure Applications

- **Property of Similar Triangles**
  - If ΔABC is similar to ΔXYZ, then their corresponding angle measures are equal and their corresponding sides are in the same ratio.
- **Problem Solving Strategy for Geometry Applications**

Step 1. **Read** the problem and make sure all the words and ideas are understood. Draw the figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

8.1 Section 8.1 Simplify Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

513. $\frac{2a+1}{3a-2}$

514. $\frac{b-3}{b^2-16}$

515. $\frac{3xy^2}{5y}$

516. $\frac{u-3}{u^2-u-30}$

Evaluate Rational Expressions

In the following exercises, evaluate the rational expressions for the given values.

517. $\frac{4p-1}{p^2+5}$ when $p=-1$

518. $\frac{q^2-5}{q+3}$ when $\text{q}=7$

519. $\frac{y^2-8}{y^2-y-2}$ when $y=1$

520. $\frac{z^2+2}{4z-z^2}$ when $\text{z}=3$

Simplify Rational Expressions

In the following exercises, simplify.

521. $\frac{10}{24}$

522. $\frac{8m^4}{16mn^3}$

523. $\frac{14a-14}{a-1}$

524. $\frac{b^2+7b+12}{b^2+8b+16}$

Simplify Rational Expressions with Opposite Factors

In the following exercises, simplify.

525. $\frac{c^2 - c - 2}{4 - c^2}$

526. $\frac{d - 16}{16 - d}$

527. $\frac{7v - 35}{25 - v^2}$

528. $\frac{w^2 - 3w - 28}{49 - w^2}$

8.2 Section 8.2 Multiply and Divide Rational Expressions

Multiply Rational Expressions

In the following exercises, multiply.

529. $\frac{3}{8} \cdot \frac{2}{15}$

530. $\frac{2xy^2}{8y^3} \cdot \frac{16y}{24x}$

531. $\frac{3a^2 + 21a}{a^2 + 6a - 7} \cdot \frac{a - 1}{ab}$

532. $\frac{5z^2}{5z^2 + 40z + 35} \cdot \frac{z^2 - 1}{3z}$

Divide Rational Expressions

In the following exercises, divide.

533. $\frac{t^2 - 4t + 12}{t^2 + 8t + 12} \div \frac{t^2 - 36}{6t}$

534. $\frac{r^2 - 16}{4} \div \frac{r^3 - 64}{2r^2 - 8r + 32}$

535. $\frac{11 + w}{w - 9} \div \frac{121 - w^2}{9 - w}$

536. $\frac{3y^2 - 12y - 63}{4y + 3} \div (6y^2 - 42y)$

537. $\frac{\frac{c^2 - 64}{3c^2 + 26c + 16}}{\frac{c^2 - 4c - 32}{15c + 10}}$

538. $\frac{8m^2 - 8m}{m - 4} \cdot \frac{m^2 + 2m - 24}{m^2 + 7m + 10} \div \frac{2m^2 - 6m}{m + 5}$

8.3 Section 8.3 Add and Subtract Rational Expressions with a Common Denominator

Add Rational Expressions with a Common Denominator

In the following exercises, add.

539. $\frac{3}{5} + \frac{2}{5}$

540. $\frac{4a^2}{2a - 1} - \frac{1}{2a - 1}$

541. $\frac{p^2 + 10p}{p + 5} + \frac{25}{p + 5}$

542. $\frac{3x}{x - 1} + \frac{2}{x - 1}$

Subtract Rational Expressions with a Common Denominator

In the following exercises, subtract.

543. $\frac{d^2}{d + 4} - \frac{3d + 28}{d + 4}$

544. $\frac{z^2}{z + 10} - \frac{100}{z + 10}$

545. $\frac{4q^2 - q + 3}{q^2 + 6q + 5} - \frac{3q^2 - q - 6}{q^2 + 6q + 5}$

546. $\frac{5t + 4t + 3}{t^2 - 25} - \frac{4t^2 - 8t - 32}{t^2 - 25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add and subtract.

547. $\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$

548. $\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$

549. $\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$

550. $\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$

8.4 Section 8.4 Add and Subtract Rational Expressions With Unlike Denominators

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

551. $\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}$

552. $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

553. $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2-23p-8}$

Find Equivalent Rational Expressions

In the following exercises, rewrite as equivalent rational expressions with the given denominator.

554. Rewrite as equivalent rational expressions with denominator $(m+2)(m-5)(m+4)$: $\frac{4}{m^2-3m-10}, \frac{2m}{m^2-m-20}$.

555. Rewrite as equivalent rational expressions with denominator $(n-2)(n-2)(n+2)$: $\frac{6}{n^2-4n+4}, \frac{2n}{n^2-4}$.

556. Rewrite as equivalent rational expressions with denominator $(3p+1)(p+6)(p+8)$: $\frac{5}{3p^2+19p+6}, \frac{7p}{3p^2+25p+8}$

Add Rational Expressions with Different Denominators

In the following exercises, add.

557. $\frac{2}{3}+\frac{3}{5}$

558. $\frac{7}{5a}+\frac{3}{2b}$

559. $\frac{2}{c-2}+\frac{9}{c+3}$

560. $\frac{3d}{d^2-9}+\frac{5}{d^2+6d+9}$

561. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

562. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

Subtract Rational Expressions with Different Denominators

In the following exercises, subtract and add.

563. $\frac{3v}{v+2}-\frac{v+2}{v+8}$

564. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

565. $\frac{7m+3}{m+2}-5$

566. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

567. $\frac{8d}{d^2-64}-\frac{4}{d+8}$

568. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

8.5 Section 8.5 Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify.

569. $\frac{\frac{5a}{a+2}}{\frac{10a^2}{a^2-4}}$

570. $\frac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

571. $\frac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

572. $\dfrac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

573. $\dfrac{6+\frac{2}{q-4}}{\frac{5}{q+4}}$

574. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

575. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

576. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

8.6 Section 8.6 Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

577. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

578. $1-\frac{2}{m}=\frac{8}{m^2}$

579. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

580. $\frac{3}{q+8}-\frac{2}{q-2}=1$

581. $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

582. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

583. $\frac{V}{l}=hw$ for l

584. $\frac{1}{x}-\frac{2}{y}=5$ for y

585. $x=\frac{y+5}{z-7}$ for z

586. $P=\frac{k}{V}$ for V

8.7 Section 8.7 Solve Proportion and Similar Figure Applications Similarity

Solve Proportions

In the following exercises, solve.

587. $\frac{x}{4}=\frac{3}{5}$

588. $\frac{3}{y}=\frac{9}{5}$

589. $\frac{s}{s+20}=\frac{3}{7}$

590. $\frac{t-3}{5}=\frac{t+2}{9}$

In the following exercises, solve using proportions.

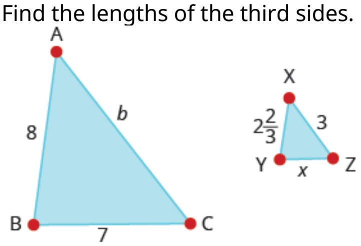
591. Rachael had a 21 ounce strawberry shake that has 739 calories. How many calories are there in a 32 ounce shake?

592. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

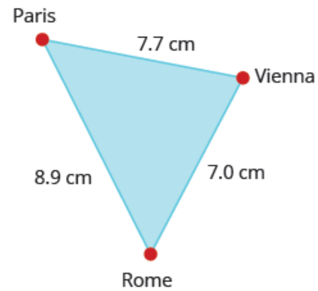
593. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



594. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

ⓐ Paris to Rome

ⓑ Paris to Vienna



595. Tony is 5.75 feet tall. Late one afternoon, his shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

596. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5 foot tall Natalie cast a 1.1 foot shadow How long would the shadow of the lighthouse be?

8.8 Section 8.8 Solve Uniform Motion and Work Applications Problems

Solve Uniform Motion Applications

In the following exercises, solve.

597. When making the 5-hour drive home from visiting her parents, Lisa ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles in the bad weather. How fast did she drive when the weather was bad?

598. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

599. John can ride his bicycle 8 mph faster than Luke can ride his bike. It takes Luke 3 hours longer than John to ride 48 miles. How fast can John ride his bike?

600. Mark was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

601. Jerry can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

602. Lisa takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

603. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

604. Sue and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

8.9 Section 8.9 Use Direct and Inverse Variation

Solve Direct Variation Problems

In the following exercises, solve.

605. If y varies directly as x, when $y = 9$ and $x = 3$, find x when $y = 21.$

606. If y varies inversely as x, when $y = 20$ and $x = 2$ find y when $x = 4.$

607. If m varies inversely with the square of n, when $m = 4$ and $n = 6$ find m when $n = 2.$

608. Vanessa is traveling to see her fiancé. The distance, *d*, varies directly with the speed, *v*, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

609. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

610. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

611. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

612. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12 inch string have?

PRACTICE TEST

In the following exercises, simplify.

613. $\dfrac{3a^2b}{6ab^2}$

614. $\dfrac{5b-25}{b^2-25}$

In the following exercises, perform the indicated operation and simplify.

615. $\dfrac{4x}{x+2}\cdot\dfrac{x^2+5x+6}{12x^2}$

616. $\dfrac{5y}{4y-8}\cdot\dfrac{y^2-4}{10}$

617. $\dfrac{4}{pq}+\dfrac{5}{p}$

618. $\dfrac{1}{z-9}-\dfrac{3}{z+9}$

619. $\dfrac{\frac{2}{3}+\frac{3}{5}}{\frac{2}{5}}$

620. $\dfrac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

In the following exercises, solve each equation.

621. $\dfrac{1}{2}+\dfrac{2}{7}=\dfrac{1}{x}$

622. $\dfrac{5}{y-6}=\dfrac{3}{y+6}$

623. $\dfrac{1}{z-5}+\dfrac{1}{z+5}=\dfrac{1}{z^2-25}$

624. $\dfrac{t}{4}=\dfrac{3}{5}$

625. $\dfrac{2}{r-2}=\dfrac{3}{r-1}$

In the following exercises, solve.

626. If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

627. If y varies inversely with x and $x=6$ when $y=20$, find y when $x=2$.

628. If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

629. The recommended erythromycin dosage for dogs, is 5 mg for every pound the dog weighs. If Daisy weighs 25 pounds, how many milligrams of erythromycin should her veterinarian prescribe?

630. Julia spent 4 hours Sunday afternoon exercising at the gym. She ran on the treadmill for 10 miles and then biked for 20 miles. Her biking speed was 5 mph faster than her running speed on the treadmill. What was her running speed?

631. Kurt can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Kurt's speed on his bike?

632. Amanda jogs to the park 8 miles using one route and then returns via a 14-mile route. The return trip takes her 1 hour longer than her jog to the park. Find her jogging rate.

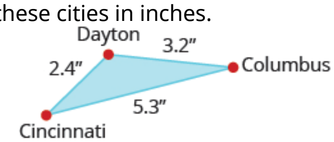
633. An experienced window washer can wash all the windows in Mike's house in 2 hours, while a new trainee can wash all the windows in 7 hours. How long would it take them working together?

634. Josh can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Josh's dad working alone to split the logs?

635. The price that Tyler pays for gas varies directly with the number of gallons he buys. If 24 gallons cost him $59.76, what would 30 gallons cost?

636. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

637. The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio, as shown on the figure below, that gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?